Properties of fuzzy set spaces with L_p metrics [☆]

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Abstract

In this paper, we discuss the properties of the spaces of fuzzy sets in a metric space with L_p-type d_p metrics, $p \geq 1$. Firstly, we give the characterizations of compactness in fuzzy set space with d_p metrics. Then we present the completions of fuzzy set spaces with d_p metrics. The d_p metrics are well-defined if and only if a function induced by the Hausdorff metric is measurable. In this paper, we give some fundamental conclusions on the measurability of this function.

Keywords: L_p metric; Hausdorff metric; Compactness; Completion; Measurability

1. Introduction

The L_p-type metrics are widely used in theoretical research and practical applications. The class of d_p metrics are a kind of L_p-type metrics. The d_p metrics are commonly used metrics on fuzzy sets [1, 2, 18, 21].

Compactness is a fundamental property in both theory and applications [6, 14, 17]. The characterizations of compactness for fuzzy set space with d_p metrics have attracted attentions of scholars [15, 22, 23]. The completion of a given metric space is an important topic in analysis.

In [7], we give the characterizations of total boundedness, relative compactness and compactness for fuzzy set space with d_p metrics. We also give the completions of various fuzzy set spaces with d_p metrics. All the fuzzy

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sets involved in the conclusions in [7] are fuzzy sets in the m-dimensional Euclidean space $\mathbb{R}^m$.

It is natural to consider spaces of fuzzy sets in a metric space [4, 5, 13]. In this paper, we discuss the properties spaces of fuzzy sets in a general metric space with d_p metrics.

We discuss the properties of the d_p^* metric (d_p metrics) and the H_{end} metric including the relationship between them. Based on these results and the characterizations of total boundedness, relative compactness and compactness in fuzzy set space with d_p metrics given in [9], We present the characterizations of total boundedness, relative compactness and compactness for space of fuzzy sets in a metric space with d_p metrics. These results reveal a connection between a set being compact in the sense of the d_p metrics and compact in the sense of the endograph metrics. The results on the characterizations of total boundedness, relative compactness and compactness in this paper generalize the corresponding results in [7]. Furthermore, using the results in this paper, we give new characterizations of total boundedness, relative compactness and compactness for space of fuzzy sets in $\mathbb{R}^m$.

We construct completions of fuzzy set spaces in a metric space with d_p metrics. These conclusions on the completions of the spaces of fuzzy set in a metric space (X, d) apply to not only the cases that X is a complete metric space but also the cases that X is an incomplete metric space. The fuzzy sets involved in the corresponding results in [7] is fuzzy sets in $\mathbb{R}^m$, which is a complete space. The results on completions of fuzzy set spaces in this paper improve the corresponding results in [7].

The d_p metrics are well-defined if and only if a function induced by the Hausdorff metric is measurable. So it is important to discuss the measurability of this function. In [8], we give some conclusions on this topic. In this paper, we give proofs to the positive conclusions and counterexamples to the negative conclusion. Further, we make great improvements to these conclusions. The conclusions on measurability of the function in this paper pointed out the cases in which the d_p metrics are well-defined, and, therefore, indicate the properties of the L_p-type metrics can be used in these cases. So these conclusions are fundamental for relevant studies of the d_p metrics.

The remainder of this paper is organized as follows. In Section 2, we recall and give some basic notions and results related to fuzzy sets and metrics on them. In Section 3, we discuss the properties and relation of the d_p^* metric and the H_{end} metric. In Section 4, we give the characterizations of total boundedness, relative compactness and compactness in $(F_{USCG}^1(X)^p, d_p)$. In

2

Section 5, we pointed out that the results in Section 4 generalize the corresponding results in [7] for $(F_{USCG}^1(\mathbb{R}^m)^p, d_p)$. Furthermore, by using results in Sections 3 and 4, we give new characterizations of total boundedness, relative compactness and compactness in $(F_{USCG}^1(\mathbb{R}^m)^p, d_p)$. In Section 6, we construct completions of spaces of fuzzy sets in a metric space. In Section 7, we give fundamental conclusions on the measurability of the function induced by the Hausdorff metric. In Section 8, we draw our conclusions.

2. Fuzzy sets and metrics on them

In this section, we recall and give some notions and results related to fuzzy sets and metrics on them. Readers can refer to [1, 2, 16, 19–21] for studies and applications of fuzzy sets.

Let $\mathbb{N}$ be the set of all natural numbers, and let $\mathbb{R}^m$ be the m-dimensional Euclidean space.

In this paper, if not specifically mentioned, we suppose that X is a metric space endowed with a metric d.

Let $K(X)$ and $C(X)$ denote the set of all non-empty compact subsets of X and the set of all non-empty closed subsets of X, respectively.

Let $F(X)$ denote the set of all fuzzy sets in X. A fuzzy set $u \in F(X)$ can be seen as a function $u : X \to [0, 1]$. A subset S of X can be seen as a fuzzy set in X. If there is no confusion, the fuzzy set in X corresponding to S is often denoted by χ_S; that is,

$$\chi_S(x) = \begin{cases} 1, & x \in S, \\ 0, & x \in X \setminus S. \end{cases}$$

For simplicity, for $x \in X$, we will use $\widehat{x}$ to denote the fuzzy set $\chi_{\{x\}}$ in X. In this paper, if we want to emphasize a specific metric space X, we will write the fuzzy set in X corresponding to S as $S_{F(X)}$, and the fuzzy set in X corresponding to $\{x\}$ as $\widehat{x}_{F(X)}$.

For $u \in F(X)$, let $[u]_\alpha$ denote the α-cut of u, i.e.

$$[u]_\alpha = \begin{cases} \{x \in X : u(x) \geq \alpha\}, & \alpha \in (0, 1], \\ \operatorname{supp} u = \overline{\{u > 0\}}, & \alpha = 0, \end{cases}$$

where $\overline{S}$ denotes the topological closure of S in (X, d).

For $u \in F(X)$, define

$$\operatorname{end} u := \{(x, t) \in X \times [0, 1] : u(x) \geq t\},$$

3

$$\operatorname{send} u := \{(x,t) \in X \times [0,1] : u(x) \geq t\} \cap ([u]_0 \times [0,1]).$$

$\operatorname{end} u$ and $\operatorname{send} u$ are called the endograph of u and the sendograph of u, respectively.

Let $F^1_{USC}(X)$ denote the set of all normal and upper semi-continuous fuzzy sets $u : X \to [0,1]$, i.e.,

$$F^1_{USC}(X) := \{u \in F(X) : [u]_\alpha \in C(X) \text{ for all } \alpha \in [0,1]\}.$$

We introduce some subclasses of $F^1_{USC}(X)$, which will be discussed in this paper. Define

$$F^1_{USCB}(X) := \{u \in F^1_{USC}(X) : [u]_0 \in K(X)\},$$
$$F^1_{USCG}(X) := \{u \in F^1_{USC}(X) : [u]_\alpha \in K(X) \text{ for all } \alpha \in (0,1]\}.$$

Let (X,d) be a metric space. We use $\boldsymbol{H}$ to denote the **Hausdorff distance** on $C(X)$ induced by d, i.e.,

$$\boldsymbol{H(U,V)} = \max\{H^*(U,V), \ H^*(V,U)\} \tag{1}$$

for arbitrary $U, V \in C(X)$, where

$$H^*(U,V) = \sup_{u \in U} d(u,V) = \sup_{u \in U} \inf_{v \in V} d(u,v).$$

We call H^* the *Hausdorff pre-distance* related to H.

The metric $\overline{d}$ on $X \times [0,1]$ is defined as

$$\overline{d}((x,\alpha),(y,\beta)) = d(x,y) + |\alpha - \beta|.$$

If there is no confusion, we also use H to denote the Hausdorff distance on $C(X \times [0,1])$ induced by $\overline{d}$.

Remark 2.1. ρ is said to be a *metric* on Y if ρ is a function from $Y \times Y$ into $\mathbb{R}$ satisfying positivity, symmetry and triangle inequality. At this time, (Y, ρ) is said to be a metric space.

ρ is said to be an *extended metric* on Y if ρ is a function from $Y \times Y$ into $\mathbb{R} \cup \{+\infty\}$ satisfying positivity, symmetry and triangle inequality. At this time, (Y, ρ) is said to be an extended metric space.

We can see that for arbitrary metric space (X,d), the Hausdorff distance H on $K(X)$ induced by d is a metric. So the Hausdorff distance H on

4

$K(X \times [0,1])$ induced by $\bar{d}$ on $X \times [0,1]$ is a metric. In these cases, we call the Hausdorff distance the Hausdorff metric.

The Hausdorff distance H on $C(X)$ induced by d on X is an extended metric, but probably not a metric, because $H(A, B)$ could be equal to $+\infty$ for certain metric space X and $A, B \in C(X)$. Clearly, if H on $C(X)$ induced by d is not a metric, then H on $C(X \times [0,1])$ induced by $\bar{d}$ is also not a metric. So the Hausdorff distance H on $C(X \times [0,1])$ induced by $\bar{d}$ on $X \times [0,1]$ is an extended metric but probably not a metric. In the cases that the Hausdorff distance H is an extended metric, we call the Hausdorff distance the Hausdorff extended metric.

We can see that H on $C(\mathbb{R}^m)$ is an extended metric but not a metric, and then the same is H on $C(\mathbb{R}^m \times [0,1])$.

In this paper, for simplicity, we refer to both the Hausdorff extended metric and the Hausdorff metric as the **Hausdorff metric**.

The d_∞ metric on $F^1_{USC}(X)$ is defined as

$$\boldsymbol{d_\infty(u, v)} := \sup\{H([u]_\alpha, [v]_\alpha) : \alpha \in [0,1]\}.$$

The endograph metric H_{end} and the sendograph metric H_{send} can be defined on $F^1_{USC}(X)$ as usual. For $u, v \in F^1_{USC}(X)$,

$$\boldsymbol{H_{\text{end}}(u, v)} := H(\text{end}\, u, \text{end}\, v),$$
$$\boldsymbol{H_{\text{send}}(u, v)} := H(\text{send}\, u, \text{send}\, v).$$

The endograph metric H_{end} and the sendograph metric H_{send} are defined by using the Hausdorff metric on $C(X \times [0,1])$ induced by $\bar{d}$ on $X \times [0,1]$.

Clearly for $u, v \in F^1_{USC}(X)$,

$$d_\infty(u, v) \geq H_{\text{send}}(u, v) \geq H_{\text{end}}(u, v).$$

Remark 2.2. We can see that H_{end} is a metric on $F^1_{USC}(X)$ with $H_{\text{end}}(u, v) \leq 1$ for all $u, v \in F^1_{USC}(X)$. Both d_∞ and H_{send} are metrics on $F^1_{USCB}(X)$. However, each one of d_∞ and H_{send} on $F^1_{USC}(X)$ is an extended metric but probably not a metric. See also Remark 3.3 in [8] (We made a misprint in the last sentence of Remark 3.3 in [8]. The "H_{end}" must be deleted from this sentence).

We can see that both d_∞ and H_{send} on $F^1_{USCG}(\mathbb{R}^m)$ are not metrics, they are extended metrics.

For simplicity, in this paper, we call H_{send} on $F^1_{USC}(X)$ the H_{send} metric or the sendograph metric H_{send}. We call d_∞ on $F^1_{USC}(X)$ the d_∞ metric or the supremum metric d_∞.

For $u, v \in F^1_{USC}(X)$, the d_p distance given by

$$d_p(u,v) = \left(\int_0^1 H([u]_\alpha, [v]_\alpha)^p \, d\alpha \right)^{1/p}$$

is well-defined if and only if $H([u]_\alpha, [v]_\alpha)$ is a measurable function of α on $[0,1]$. In the sequel, we suppose that the d_p distance is with $p \geq 1$.

In Section 7, we give some fundamental conclusions for the measurability of the function $H([u]_\alpha, [v]_\alpha)$ of α on $[0,1]$. We list some of these conclusions in the following.

(i) The function $H([u]_\alpha, [v]_\alpha)$ of α on $[0,1]$ is measurable when $u, v \in F^1_{USC}(\mathbb{R}^m)$ (Proposition 7.5). So the d_p distance is well-defined on $F^1_{USC}(\mathbb{R}^m)$.

(ii) The function $H([u]_\alpha, [v]_\alpha)$ of α on $[0,1]$ is measurable when $u \in F^1_{USC}(X)$ and $v \in F^1_{USCG}(X)$ (Theorem 7.15).

As a corollary, d_p distance is well-defined on $F^1_{USCG}(X)$ (Proposition 7.8)

As a corollary, $d_p(u, \widehat{x_0})$ is well-defined when $u \in F^1_{USC}(X)$ and x_0 is a point in X (Proposition 7.1).

(iii) In Section 7, we improve conclusions in the above clauses (i) and (ii). Corollary 7.19 is an improvement of Proposition 7.5. Corollary 7.20 is an improvement of Theorem 7.15.

Since $H([u]_\alpha, [v]_\alpha)$ could be a non-measurable function of α on $[0,1]$ (see Example 7.14), we introduce the d_p^* distance on $F^1_{USC}(X)$, $p \geq 1$, in [8], which is defined by

$$d_p^*(u,v) := \inf\{ \left(\int_0^1 f(\alpha)^p \, d\alpha \right)^{1/p} : f \text{ is a measurable function from } [0,1] \text{ to } \mathbb{R} \cup \{+\infty\};$$

$$f(\alpha) \geq H([u]_\alpha, [v]_\alpha) \text{ for } \alpha \in [0,1] \}$$

for $u, v \in F^1_{USC}(X)$.

We can see that if $d_p(u,v)$ is well-defined for $u, v \in F^1_{USC}(X)$, then $d_p^*(u,v) = d_p(u,v)$. The d_p^* distance is an expansion of the d_p distance on $F^1_{USC}(X)$.

6

Remark 2.3. In the sequel, we write $d_p(u,v)$ rather than $d_p^*(u,v)$ when $d_p(u,v)$ is well-defined.

In [8], we point out that d_p^* on $F_{USC}^1(X)$ is an extended metric but probably not a metric (See Theorem 3.1 and Remark 3.3 in [8]).

The d_p distance on $F_{USCG}^1(X)$ is an extended metric but probably not a metric because $d_p(u,v) = +\infty$ could happen for $u,v \in F_{USCG}^1(X)$. For example, define $u \in F(\mathbb{R})$ by

$$u(x) = \begin{cases} 0, & x < 1, \\ 1, & x = 1, \\ 1/n, & x \in (n^2, (n+1)^2], \ n = 1, 2, \ldots. \end{cases}$$

Then $u \in F_{USCG}^1(\mathbb{R})$, $\widehat{1}_{F(\mathbb{R})} \in F_{USCB}^1(\mathbb{R}) \subseteq F_{USCG}^1(\mathbb{R})$, and $d_p(u, \widehat{1}_{F(\mathbb{R})}) = +\infty$.

We can see that the d_p distance on $F_{USCB}^1(X)$ is a metric; the d_p distance on $F_{USCG}^1(\mathbb{R}^m)$ is an extended metric but not a metric; the d_p distance on $F_{USC}^1(\mathbb{R}^m)$ is an extended metric but not a metric.

Remark 2.4. In this paper, for simplicity, we refer to both the d_p^* extended metric and the d_p^* metric as the d_p^* metric, and both the d_p extended metric and the d_p metric as the d_p metric.

We introduce the following subset of $F_{USC}^1(X)$

- $F_{USCG}^1(X)^p := \{u \in F_{USCG}^1(X) : d_p(u, \widehat{x_0}) = (\int_0^1 H([u]_\alpha, \{x_0\})^p \, d\alpha)^{1/p} < +\infty\}$, where x_0 is a point in X.

The definition of $F_{USCG}^1(X)^p$ does not depend on the choice of x_0. We can see that the d_p distance is a metric on $F_{USCG}^1(X)^p$.

Clearly,

$$F_{USCB}^1(X) \subseteq F_{USCG}^1(X)^p \subseteq F_{USCG}^1(X) \subseteq F_{USC}^1(X).$$

We use $(\widetilde{X}, \widetilde{d})$ to denote the completion of (X,d). We see (X,d) as a subspace of $(\widetilde{X}, \widetilde{d})$. Let $S \subseteq \widetilde{X}$. The symbol $\overline{\overline{S}}$ is used to denote the closure of S in $(\widetilde{X}, \widetilde{d})$.

As defined previously, we have $K(\widetilde{X})$, $C(\widetilde{X})$, $F_{USC}^1(\widetilde{X})$, $F_{USCG}^1(\widetilde{X})$, etc. according to $(\widetilde{X}, \widetilde{d})$. For example,

$$F_{USC}^1(\widetilde{X}) := \{u \in F(\widetilde{X}) : [u]_\alpha \in C(\widetilde{X}) \text{ for all } \alpha \in [0,1]\},$$

$$F_{USCG}^1(\widetilde{X}) := \{u \in F(\widetilde{X}) : [u]_\alpha \in K(\widetilde{X}) \text{ for all } \alpha \in (0,1]\}.$$

If there is no confusion, we also use H to denote the Hausdorff metric on $C(\widetilde{X})$ induced by $\widetilde{d}$. We also use H to denote the Hausdorff metric on $C(\widetilde{X} \times [0,1])$ induced by $\overline{\widetilde{d}}$. We also use H_{end} to denote the endograph metric on $F_{USC}^1(\widetilde{X})$ given by using H on $C(\widetilde{X} \times [0,1])$. We also use d_p to denote the d_p metric on $F_{USCG}^1(\widetilde{X})$.

Define $f : F_{USCG}^1(X) \to F_{USCG}^1(\widetilde{X})$ as follows: for $u \in F_{USCG}^1(X)$,

$$f(u)(t) = \begin{cases} u(t), & t \in X, \\ 0, & t \in \widetilde{X} \setminus X. \end{cases}$$

Then $[f(u)]_\alpha = [u]_\alpha$ for all $\alpha \in (0,1]$, and so $f(u) \in F_{USCG}^1(\widetilde{X})$. We can see that for $\rho = d_\infty, d_p, H_{\text{end}}, H_{\text{send}}$, $\rho(u,v) = \rho(f(u), f(v))$. So for $\rho = d_\infty, d_p, H_{\text{send}}$, $(F_{USCG}^1(X), \rho)$ can be embedded as an extended metric subspace of $(F_{USCG}^1(\widetilde{X}), \rho)$. $(F_{USCG}^1(X), H_{\text{end}})$ can be embedded as a metric subspace of $(F_{USCG}^1(\widetilde{X}), H_{\text{end}})$.

In this paper, we see $(F_{USCG}^1(X), H_{\text{end}})$ as a metric subspace of $(F_{USCG}^1(\widetilde{X}), H_{\text{end}})$. We see $(F_{USCG}^1(X)^p, d_p)$ as a metric subspace of $(F_{USCG}^1(\widetilde{X})^p, d_p)$. We see $(F_{USCG}^1(X), d_p)$ as an extended metric subspace of $(F_{USCG}^1(\widetilde{X}), d_p)$.

3. Properties of d_p^* metric and H_{end} metric

In this section, we give the relationship between the d_p^* metric and the H_{end} metric. We illustrate the relations between the property that U is uniformly p-mean bounded and other properties of U. We list the properties of $(F_{USCG}^1(X), H_{\text{end}})$ obtained in [12] which is useful in this paper.

Theorem 3.1. *Let (X,d) be a metric space.*
(i) For $u, v \in F_{USC}^1(X)$,

$$d_p^*(u,v) \geq \left(\frac{H_{\text{end}}(u,v)^{p+1}}{p+1}\right)^{1/p}. \tag{2}$$

(ii) For $u \in F_{USC}^1(X)$ and a sequence $\{u_n\}$ in $F_{USC}^1(X)$, if $d_p^*(u_n, u) \to 0$, then $H_{\text{end}}(u_n, u) \to 0$.

8

Proof. To show (i), we only need to show that for each $r > 0$, if $H_{\text{end}}(u, v) > r$ then $d_p^*(u, v) \geq \left(\frac{r^{p+1}}{p+1}\right)^{1/p}$.

Let $r > 0$. Assume that $H_{\text{end}}(u, v) > r$. Then without loss of generality we suppose that $H^*(\text{end}\, u, \text{end}\, v) > r$, then there is an $(x, \beta) \in \text{end}\, u$, such that $\overline{d}((x, \beta), \text{end}\, v) > r$. This implies that $\beta > r$ and $d(x, [v]_\alpha) > r - (\beta - \alpha)$ when $\alpha \in [\beta - r, \beta]$. Hence $H^*([u]_\alpha, [v]_\alpha) > r - (\beta - \alpha)$ when $\alpha \in [\beta - r, \beta]$.

Let f be a measurable function on $[0, 1]$ with $f(\alpha) \geq H([u]_\alpha, [v]_\alpha)$ for $\alpha \in [0, 1]$. Then

$$
\left(\int_0^1 f(\alpha)^p \, d\alpha\right)^{1/p} \geq \left(\int_{\beta-r}^\beta f(\alpha)^p \, d\alpha\right)^{1/p}
$$
$$
> \left(\int_{\beta-r}^\beta (r - (\beta - \alpha))^p \, d\alpha\right)^{1/p}
$$
$$
= \left(\frac{r^{p+1}}{p+1}\right)^{1/p}.
$$

So from the definition of $d_p^*(u, v)$, we have $d_p^*(u, v) \geq \left(\frac{r^{p+1}}{p+1}\right)^{1/p}$.

(ii) follows immediately from (i).

$\square$

Remark 3.2. Clearly, if $d_p(u, v)$ is well-defined (at this time $d_p^*(u, v) = d_p(u, v)$), then $d_p^*(u, v)$ can be replaced by $d_p(u, v)$ in Theorem 3.1.

The "=" can be obtained in (2).

Example 3.3. Define u and v in $F_{USCB}^1(\mathbb{R})$ as

$$
u(x) = \begin{cases} 1, & x = 0, \\ 0.5 - x, & x \in (0, 0.5], \\ 0, & \text{otherwise}, \end{cases} \quad v(x) = \begin{cases} 1, & x = 0, \\ 0.5, & x \in (0, 0.5], \\ 0, & \text{otherwise}. \end{cases}
$$

Then $H_{\text{end}}(u, v) = 0.5$ and

$$
H([u]_\alpha, [v]_\alpha) = \begin{cases} 0, & \alpha \in (0.5, 1], \\ \alpha, & \alpha \in [0, 0.5]. \end{cases}
$$

Thus $d_p(u, v) = (\int_0^{0.5} \alpha^p \, d\alpha)^{1/p} = \left(\frac{0.5^{p+1}}{p+1}\right)^{1/p} = \left(\frac{H_{\text{end}}(u,v)^{p+1}}{p+1}\right)^{1/p}$.

The metric $\widehat{d}$ on $X \times [0,1]$ is defined as

$$\widehat{d}((x,\alpha),(y,\beta)) = \max\{d(x,y), |\alpha - \beta|\}.$$

We can see that for $(x,\alpha), (y,\beta)$ in $X \times [0,1]$,

$$\begin{aligned}
\widehat{d}((x,\alpha),(y,\beta)) &\le \overline{d}((x,\alpha),(y,\beta)) \\
&\le \min\{2\widehat{d}((x,\alpha),(y,\beta)),\ d(x,y)+1\} \\
&\le \min\{2\widehat{d}((x,\alpha),(y,\beta)),\ \widehat{d}((x,\alpha),(y,\beta))+1\}.
\end{aligned} \tag{3}$$

By (3), $\widehat{d}$ induce the same topology on $X \times [0,1]$ as $\overline{d}$. In this paper, $C(X \times [0,1])$ is used to denote the set of all non-empty closed subsets of $(X \times [0,1], \overline{d})$. $C(X \times [0,1])$ is also the set of all non-empty closed subsets of $(X \times [0,1], \widehat{d})$.

We use H' to denote the Hausdorff extended metric on $C(X \times [0,1])$ induced by $\widehat{d}$ on $X \times [0,1]$.

The endograph metric H'_{end} and the sendograph extended metric H'_{send} can be defined on $F^1_{USC}(X)$ by using the Hausdorff extended metric H' on $C(X \times [0,1])$ as follows. For $u, v \in F^1_{USC}(X)$,

$$\begin{aligned}
\boldsymbol{H'_{\mathrm{end}}(u,v)} &:= H'(\mathrm{end}\, u, \mathrm{end}\, v), \\
\boldsymbol{H'_{\mathrm{send}}(u,v)} &:= H'(\mathrm{send}\, u, \mathrm{send}\, v).
\end{aligned}$$

Remark 3.4. We can see that each one of H' on $C(X \times [0,1])$ and H'_{send} on $F^1_{USC}(X)$ is an extended metric but probably not a metric.

We can see that both H' on $C(\mathbb{R}^m \times [0,1])$ and H'_{send} on $F^1_{USCG}(\mathbb{R}^m)$ are not metrics, they are extended metrics.

For simplicity, in this paper, we call the Hausdorff extended metric H' on $C(X \times [0,1])$ the Hausdorff metric H', and call H'_{send} on $F^1_{USC}(X)$ the H'_{send} metric or the sendograph metric H'_{send}.

In some references, the sendograph metric and the endograph metric refer to H'_{send} and H'_{end}, respectively. In the following, we give some conclusions on H'_{send} and H'_{end}.

By Proposition 3.6 below, we can see that for u and u_n, $n = 1,2,\ldots$, in $F^1_{USC}(X)$, (i) $H'_{\mathrm{send}}(u_n, u) \to 0$ if and only if $H_{\mathrm{send}}(u_n, u) \to 0$, (ii) $H'_{\mathrm{end}}(u_n, u) \to 0$ if and only if $H_{\mathrm{end}}(u_n, u) \to 0$. Our purpose of giving Proposition 3.6 is to give the above conclusion.

10

Clearly for $u, v \in F^1_{USC}(X)$,

$$d_\infty(u,v) \geq H'_{\text{send}}(u,v) \geq H'_{\text{end}}(u,v).$$

Proposition 3.7 and Example 3.9 below illustrate the relationship between d^*_p and H'_{end} on $F^1_{USC}(X)$. Proposition 3.7 and Example 3.9 are parallel conclusions of Theorem 3.1 and Example 3.3, respectively.

We use $\mathbb{R}^+$ to denote the set $\{x \in \mathbb{R} : x \geq 0\}$. We say a function $f : \mathbb{R}^+ \to \mathbb{R}^+$ is increasing if $f(x) \leq f(y)$ when $x < y$.

Proposition 3.5. *Let f be an increasing and continuous function from $\mathbb{R}^+$ to $\mathbb{R}^+$. Let Z be a set and let d_1 and d_2 be two metrics on Z. If $d_1(\xi, \eta) \leq f(d_2(\xi, \eta))$ for all $\xi, \eta \in Z$, then for each $S \subseteq Z$ and each $W \subseteq Z$,*
(i) $d_1(\xi, S) \leq f(d_2(\xi, S))$,
(ii) $H^*_1(W, S) \leq f(H^*_2(W, S))$, and
(iii) $H_1(W, S) \leq f(H_2(W, S))$,
where H_1 and H_2 are the Hausdorff extended metrics induced by d_1 and d_2, respectively, H^*_1 and H^*_2 are the Hausdorff pre-distance related to H_1 and H_2, respectively.

Proof. (i) is true because

$$d_1(\xi, S) = \inf_{\eta \in S} d_1(\xi, \eta) \leq \inf_{\eta \in S} f(d_2(\xi, \eta)) = f(\inf_{\eta \in S} d_2(\xi, \eta)) = f(d_2(\xi, S)).$$

From (i), we can obtain (ii) is true since

$$H^*_1(W, S) = \sup_{\xi \in W} d_1(\xi, S) \leq \sup_{\xi \in W} f(d_2(\xi, S)) = f(\sup_{\xi \in W} d_2(\xi, S)) = f(H^*_2(W, S)).$$

By (ii),

$$
\begin{aligned}
H_1(W, S) &= H^*_1(W, S) \vee H^*_1(S, W) \\
&\leq f(H^*_2(W, S)) \vee f(H^*_2(S, W)) \\
&= f(H^*_2(W, S) \vee H^*_2(S, W)) = f(H_2(W, S)).
\end{aligned}
$$

So (iii) is true.

$\square$

Proposition 3.6. *Let (X, d) be a metric space and $u, v \in F^1_{USC}(X)$. Then*

$$
\begin{aligned}
H'_{\text{send}}(u,v) &\leq H_{\text{send}}(u,v) \leq \min\{2H'_{\text{send}}(u,v),\ H'_{\text{send}}(u,v) + 1\}, \\
H'_{\text{end}}(u,v) &\leq H_{\text{end}}(u,v) \leq \min\{2H'_{\text{end}}(u,v),\ 1\}.
\end{aligned}
$$

11

Proof. Define functions f_1 and f_2 from $\mathbb{R}^+$ to $\mathbb{R}^+$ given by $f_1(x) = x$ and $f_2(x) = \min\{2x, x+1\}$. Then f_1 and f_2 are increasing and continuous functions.

By (3), for all $(x, \alpha), (y, \beta)$ in $X \times [0,1]$, $\widehat{d}((x,\alpha),(y,\beta)) \leq f_1(\overline{d}((x,\alpha),(y,\beta)))$ and $\overline{d}((x,\alpha),(y,\beta)) \leq f_2(\widehat{d}((x,\alpha),(y,\beta)))$. Then from (iii) of Proposition 3.5 and the fact that $H_{\text{end}}(u,v) \leq 1$ for $u, v \in F^1_{USC}(X)$, we can obtain that for $u, v \in F^1_{USC}(X)$,

$$H'_{\text{send}}(u,v) \leq H_{\text{send}}(u,v) \leq \min\{2H'_{\text{send}}(u,v), \ H'_{\text{send}}(u,v)+1\},$$

and

$$H'_{\text{end}}(u,v) \leq H_{\text{end}}(u,v) \leq \min\{2H'_{\text{end}}(u,v), \ H'_{\text{end}}(u,v)+1, \ 1\} = \min\{2H'_{\text{end}}(u,v), \ 1\}.$$

$\square$

The following Proposition 3.7 is proved in a similar fashion to Theorem 3.1, Example 3.9 is constructed in a similar fashion to Example 3.3.

Proposition 3.7. *Let (X,d) be a metric space.*
(i) For $u, v \in F^1_{USC}(X)$,

$$d^*_p(u,v) \geq H'_{\text{end}}(u,v)^{1+1/p}. \tag{4}$$

(ii) For $u \in F^1_{USC}(X)$ and a sequence $\{u_n\}$ in $F^1_{USC}(X)$, if $d^*_p(u_n, u) \to 0$, then $H'_{\text{end}}(u_n, u) \to 0$.

Proof. To show (i), we only need to show that for each $r > 0$, if $H'_{\text{end}}(u,v) > r$ then $d^*_p(u,v) \geq r^{1+1/p}$.

Let $r > 0$. Assume $H'_{\text{end}}(u,v) > r$. Then without loss of generality we can suppose that $H'^*(\text{end}\,u, \text{end}\,v) > r$, where H'^* denotes the Hausdorff pre-distance related to H'. Thus there is an $(x, \beta) \in \text{end}\,u$, such that $\widehat{d}((x,\beta), \text{end}\,v) > r$. This implies that $\beta > r$ and $d(x, [v]_\alpha) > r$ when $\alpha \in [\beta - r, \beta]$. Hence $H^*([u]_\alpha, [v]_\alpha) > r$ when $\alpha \in [\beta - r, \beta]$.

Let f be a measurable function on $[0,1]$ with $f(\alpha) \geq H([u]_\alpha, [v]_\alpha)$ for $\alpha \in [0,1]$. Then

$$\left(\int_0^1 f(\alpha)^p \, d\alpha\right)^{1/p} \geq \left(\int_{\beta-r}^\beta f(\alpha)^p \, d\alpha\right)^{1/p}$$

12

$$> \left(\int_{\beta-r}^{\beta} r^p \, d\alpha \right)^{1/p}$$
$$= r^{1+1/p}.$$

So from the definition of $d_p^*(u,v)$, we have $d_p^*(u,v) \geq r^{1+1/p}$.

(ii) follows immediately from (i).

$\square$

Remark 3.8. Clearly, if $d_p(u,v)$ is well-defined (at this time $d_p^*(u,v) = d_p(u,v)$), then $d_p^*(u,v)$ can be replaced by $d_p(u,v)$ in Proposition 3.7.

The "=" can be obtained in (4).

Example 3.9. Define u and v in $F_{USCB}^1(\mathbb{R})$ as

$$u(x) = \begin{cases} 1, & x = 0, \\ 0.5, & x \in (0, 0.5], \\ 0, & \text{otherwise}, \end{cases} \quad v(x) = \begin{cases} 1, & x = 0.5, \\ 0.5, & x \in [0, 0.5), \\ 0, & \text{otherwise}. \end{cases}$$

Then $H'_{\text{end}}(u,v) = 0.5$ and

$$H([u]_\alpha, [v]_\alpha) = \begin{cases} 0.5, & \alpha \in (0.5, 1], \\ 0, & \alpha \in [0, 0.5]. \end{cases}$$

Thus $d_p(u,v) = (\int_{0.5}^1 0.5^p \, d\alpha)^{1/p} = 0.5^{1+1/p} = H'_{\text{end}}(u,v)^{1+1/p}$.

The following two concepts are essentially proposed by Diamond and Kloeden [1] and Ma [15], respectively.

For $u \in F_{USCG}^1(X)$ and $h \in [0,1)$, the function $H([u]_\alpha, [u]_{\alpha-h})$ of α is left-continuous on $(h, 1]$, and thus is measurable on $[h, 1]$ (see Proposition 7.9). So $\left(\int_h^1 H([u]_\alpha, [u]_{\alpha-h})^p \, d\alpha \right)^{1/p}$ is well-defined.

Definition 3.10. *[1] Let $u \in F_{USCG}^1(X)^p$. If for given $\varepsilon > 0$, there is a $\delta(u, \varepsilon) \in (0, 1]$ such that for all $0 \leq h < \delta$*

$$\left(\int_h^1 H([u]_\alpha, [u]_{\alpha-h})^p \, d\alpha \right)^{1/p} < \varepsilon,$$

where $1 \leq p < +\infty$, then we say u is p-mean left-continuous.

Suppose that U is a nonempty set in $F_{USCG}^1(X)^p$. If the above inequality holds uniformly for all $u \in U$, then we say U is p-mean equi-left-continuous.

13

Definition 3.11. *[15] A set U in $F_{USCG}^1(X)^p$ is said to be uniformly p-mean bounded if there exist a constant $M > 0$ and an $x_0 \in X$ such that $d_p(u, x_0) \le M$ for all $u \in U$.*

We say that a set U is bounded in a metric space (Y, ρ) if and only if there is an $M > 0$ such that $\sup\{\rho(x, y) : x, y \in U\} \le M$.

Clearly, U in $F_{USCG}^1(X)^p$ is uniformly p-mean bounded is equivalent to U is bounded in $(F_{USCG}^1(X)^p, d_p)$.

Remark 3.12. Corollary 7.4 states the following conclusion:
Let $[\mu, \nu]$ be an interval in $\mathbb{R}$ and let f be a function from $[\mu, \nu]$ to $\widehat{\mathbb{R}}$, where $\widehat{\mathbb{R}} = \mathbb{R} \cup \{-\infty\} \cup \{+\infty\}$. If f is left-continuous at $\alpha \in (\mu, \nu]$, i.e., $f(\alpha) = \lim_{\gamma \to \alpha-} f(\gamma)$, then f is a measurable function on $[\mu, \nu]$.

In [8], we give Proposition 7.6. Using Proposition 7.6 (i) and the triangle inequality of the Hausdorff metric, we can show that many of the Hausdorff distance functions appearing in this paper are left-continuous on an interval $(\mu, \nu]$ and therefore measurable on $[\mu, \nu]$ by Corollary 7.4. The proofs of these conclusions are similar to that of Proposition 7.7 (i) and Proposition 7.8. In the following, we list some of such Hausdorff distance functions.

(i) For $u, v \in F_{USCG}^1(X)$, the function $H([u]_\alpha, [v]_\alpha)$ of α is left-continuous on $(0, 1]$ and thus measurable on $[0, 1]$ (see Proposition 7.7 (i) and Proposition 7.8).

(ii) For $u \in F_{USCG}^1(X)$, $h \in [0, 1)$, $H([u]_\alpha, [u]_{\alpha-h})$ is left-continuous at $\alpha \in (h, 1]$ and thus measurable on $[h, 1]$; $H([u]_\alpha, [u]_{\alpha+h})$ is left-continuous at $\alpha \in (0, 1 - h]$ and thus measurable on $[0, 1 - h]$. (see Proposition 7.9).

(iii) $H([u]_\alpha, [u]_{\alpha-\frac{1}{N}h})$ appearing in the proof of Proposition 3.16 is left-continuous on $(h - \frac{k}{N}h, 1 - \frac{k}{N}h]$, and thus measurable on $[h - \frac{k}{N}h, 1 - \frac{k}{N}h]$ for $k = 0, \ldots, N-1$. So the formula $\sum_{k=0}^{N-1} \left(\int_{h - \frac{k}{N}h}^{1 - \frac{k}{N}h} H([u]_\alpha, [u]_{\alpha-\frac{1}{N}h})^p \, d\alpha \right)^{1/p}$ appearing in the proof of Proposition 3.16 is well-defined.

Let f be a function from $[\mu, \nu]$ to $\widehat{\mathbb{R}}$ and $[\xi, \eta]$ a subinterval of $[\mu, \nu]$. If f is left-continuous on $(\mu, \nu]$, then clearly the restriction of f to $[\xi, \eta]$, denoted by $f|_{[\xi, \eta]}$, is left-continuous on $(\xi, \eta]$, and thus measurable on $[\xi, \eta]$.

14

We can also show the measurability of $f|_{[\xi,\eta]}$ as follows. Since f is measurable on $[\mu,\nu]$, then for each measurable subset S of $[\mu,\nu]$, $f|_S$ is measurable on S, and so $f|_{[\xi,\eta]}$ is measurable on $[\xi,\eta]$ because $[\xi,\eta]$ is measurable.

By using the above conclusion, we can obtain the left-continuity of a Hausdorff distance function. For instance, from the above clause (ii) or Proposition 7.9, we have that $H([u]_\alpha,[u]_{\alpha-\frac{1}{N}h})$ is left-continuous on $(h/N,1]$ and thus measurable on $[h/N,1]$. Since $[h-\frac{k}{N}h,1-\frac{k}{N}h] \subseteq [h/N,1]$ for $k = 0,\ldots,N-1$, thus we obtain the conclusion of the above clause (iii). In our opinion, this way to prove clause (iii) is essentially the same as the original way, which using Proposition 7.6 (i) and Corollary 7.4 directly.

We think Corollary 7.4 is an already known conclusion, although we can't find this conclusion in the references that we can obtain. In this sense, conclusions of the above clauses (i)-(iii) can be seen as corollaries of Proposition 7.6.

The discussions of this paper involve a kind of Hausdorff distance functions $f : [\mu,\nu] \to \widehat{\mathbb{R}}$ with the property that f is left-continuous on $(\mu,\nu]$. Thus f is measurable on $[\mu,\nu]$, and so $(\int_\mu^\nu f(\alpha)^p \, d\alpha)^{1/p}$ is well-defined. Some of these Hausdorff distance functions f are listed in the above clauses (i)-(iii). In the sequel, we will not explain the well-definedness of these $(\int_\mu^\nu f(\alpha)^p \, d\alpha)^{1/p}$ one by one.

Let U be a set in $F^1_{USCG}(X)^p$. The following Lemma 3.13 and Theorem 3.14 illustrate the relations between the property that U is uniformly p-mean bounded and other properties of U.

Lemma 3.13. *Let U be a subset of $F^1_{USCG}(X)^p$. If U is uniformly p-mean bounded, then for each $h \in (0,1]$, $U(h)$ is bounded in (X,d).*

Proof. We proceed by contradiction. Assume that there is an $h_0 \in (0,1]$ such that $U(h_0)$ is not bounded in (X,d). This means that $\sup\{H([u]_{h_0},\{x_0\}) : u \in U\} = +\infty$. Note that for each $u \in U$,

$$\left(\int_0^1 H([u]_\alpha,\{x_0\})^p \, d\alpha \right)^{1/p} \geq \left(\int_0^{h_0} H([u]_\alpha,\{x_0\})^p \, d\alpha \right)^{1/p} \geq h_0 \cdot H([u]_{h_0},\{x_0\}).$$

Thus $\sup\left\{ \left(\int_0^1 H([u]_\alpha,\{x_0\})^p \, d\alpha \right)^{1/p} : u \in U \right\} = +\infty$, which contradicts the assumption that U is uniformly p-mean bounded.

$\square$

15

Theorem 3.14. *Let U be a subset of $F^1_{USCG}(X)^p$. If U is p-mean equi-left-continuous, then the following three properties are equivalent:*
(i) There exists an $h \in (0,1)$ such that $U(h)$ is bounded in (X,d);
(ii) For each $h \in (0,1]$, $U(h)$ is bounded in (X,d);
(iii) U is uniformly p-mean bounded.

Proof. (i)$\Rightarrow$(iii). Assume that (i) is true, i.e. there is an $h_1 \in (0,1)$ such that $U(h_1)$ is bounded in (X,d). Then there exists an $L > 0$ such that $\sup\{d(x,x_0) : x,y \in U(h_1)\} \leq L$. Put $M = L \cdot (1-h_1)^{1/p}$. Then for all $h \in [h_1, 1]$ and $u \in U$,

$$\left(\int_h^1 H([u]_\alpha, \{x_0\})^p \, d\alpha \right)^{1/p} \leq L \cdot (1-h_1)^{1/p} = M. \tag{5}$$

Since U is p-mean equi-left-continuous, there is an $h_2 > 0$ such that for all $h \in [0, h_2]$ and $u \in U$,

$$\left(\int_h^1 H([u]_\alpha, [u]_{\alpha-h})^p \, d\alpha \right)^{1/p} < 1 \tag{6}$$

Choose an $h \leq \min\{1-h_1, h_2\}$ satisfying $1/h = N \in \mathbb{N}$. Then by (6) for $k = 1, \ldots, N-1$ and $u \in U$,

$$\left| \left(\int_{kh}^{(k+1)h} H([u]_\alpha, \{x_0\})^p \, d\alpha \right)^{1/p} - \left(\int_{(k-1)h}^{kh} H([u]_\alpha, \{x_0\})^p \, d\alpha \right)^{1/p} \right|$$

$$= \left| \left(\int_{kh}^{(k+1)h} H([u]_\alpha, \{x_0\})^p \, d\alpha \right)^{1/p} - \left(\int_{kh}^{(k+1)h} H([u]_{\alpha-h}, \{x_0\})^p \, d\alpha \right)^{1/p} \right|$$

$$\leq \left(\int_{kh}^{(k+1)h} H([u]_\alpha, [u]_{\alpha-h})^p \, d\alpha \right)^{1/p} \leq \left(\int_h^1 H([u]_\alpha, [u]_{\alpha-h})^p \, d\alpha \right)^{1/p} < 1,$$
$$\tag{7}$$

and thus by (5) and (7), for all $u \in U$,

$$\left(\int_0^1 H([u]_\alpha, \{x_0\})^p \, d\alpha \right)^{1/p}$$

16

$$\leq \sum_{k=0}^{N-1} \left(\int_{kh}^{(k+1)h} H([u]_\alpha, \{x_0\})^p \, d\alpha \right)^{1/p}$$
$$< M + \cdots + (M + (N-1))$$
$$= N \cdot M + N(N-1)/2,$$

and so (iii) is true.

(iii)$\Rightarrow$(ii) follows from Lemma 3.13.

(ii)$\Rightarrow$(i) is obvious.

$\square$

Remark 3.15. From Corollary 5.1 and Theorem 5.5, we know that for a p-mean equi-left-continuous set U in $F_{USCG}^1(\mathbb{R}^m)^p$, the properties (i) $U(\alpha)$ is bounded in $\mathbb{R}^m$ for each $\alpha \in (0,1]$, and (ii) U is uniformly p-mean bounded, are equivalent.

Proposition 3.16. *Let U be a subset of $F_{USCG}^1(X)^p$. If U is p-mean equi-left-continuous, then for each $h \in [0,1]$, there exists a $C_h \in \mathbb{R}$ such that for all $u \in U$,*

$$\left(\int_h^1 H([u]_\alpha, [u]_{\alpha-h})^p \, d\alpha \right)^{1/p} \leq C_h.$$

Proof. Since U is p-mean equi-left-continuous, then there is an $h_0 > 0$ such that for all $u \in U$ and $h \in [0, h_0]$,

$$\left(\int_h^1 H([u]_\alpha, [u]_{\alpha-h})^p \, d\alpha \right)^{1/p} \leq 1. \tag{8}$$

Let $h \in [0,1]$. If $h \in [0, h_0]$, then the desired result follows from (8).

If $h \in (h_0, 1]$, then there is an $N(h) \in \mathbb{N}$ such that $h/N \leq h_0$. Thus for all $u \in U$,

$$\left(\int_h^1 H([u]_\alpha, [u]_{\alpha-h})^p \, d\alpha \right)^{1/p}$$
$$\leq \sum_{k=0}^{N-1} \left(\int_h^1 H([u]_{\alpha-\frac{k}{N}h}, [u]_{\alpha-\frac{k+1}{N}h})^p \, d\alpha \right)^{1/p}$$
$$\leq \sum_{k=0}^{N-1} \left(\int_{h-\frac{k}{N}h}^{1-\frac{k}{N}h} H([u]_\alpha, [u]_{\alpha-\frac{1}{N}h})^p \, d\alpha \right)^{1/p}$$

17

$$\leq \sum_{k=0}^{N-1} \left(\int_{\frac{1}{N}h}^{1} H([u]_\alpha, [u]_{\alpha-\frac{1}{N}h})^p \, d\alpha \right)^{1/p}$$
$$\leq N.$$

Here we mention that $H([u]_{\alpha-\frac{k}{N}h}, [u]_{\alpha-\frac{k+1}{N}h})$, $k = 0, \ldots, N-1$, are left-continuous on $(h, 1]$, and thus is measurable on $[h, 1]$. This conclusion can be shown by using Proposition 7.6 and Corollary 7.4. The proof of it is similar to that of Proposition 7.7 (i) and Proposition 7.8.

We can see that

$\square$

We can see that (i)$\Rightarrow$(iii) in the proof of Theorem 3.14 can also be proved by using Proposition 3.16.

- A subset Y of a topological space Z is said to be *compact* if for every set I and every family of open sets, O_i, $i \in I$, such that $Y \subset \bigcup_{i \in I} O_i$ there exists a finite family $O_{i_1}, O_{i_2} \ldots, O_{i_n}$ such that $Y \subseteq O_{i_1} \cup O_{i_2} \cup \ldots \cup O_{i_n}$. In the case of a metric topology, the criterion for compactness becomes that any sequence in Y has a subsequence convergent in Y.

- A *relatively compact* subset Y of a topological space Z is a subset with compact closure. In the case of a metric topology, the criterion for relative compactness becomes that any sequence in Y has a subsequence convergent in Z.

- Let (X, d) be a metric space. A set U in X is *totally bounded* if and only if, for each $\varepsilon > 0$, it contains a finite ε approximation, where an ε approximation to U is a subset S of U such that $\rho(x, S) < \varepsilon$ for each $x \in U$. An ε approximation to U is also called an ε-net of U.

Let (X, d) be a metric space. A set U is compact in (X, d) implies that U is relatively compact in (X, d), which in turn implies that U is totally bounded in (X, d).

Suppose that U is a subset of $F_{USC}^1(X)$ and $\alpha \in [0, 1]$. For writing convenience, we denote

- $U(\alpha) := \bigcup_{u \in U} [u]_\alpha$, and

- $U_\alpha := \{[u]_\alpha : u \in U\}$.

18

We list the following conclusions in [9] which are on the property of H_{end} metric, the characterizations of total boundedness, relative compactness and compactness for $(F^1_{USCG}(X), H_{\text{end}})$, and the completion of $(F^1_{USCG}(X), H_{\text{end}})$. These conclusions will be useful in this paper.

Theorem 3.17. *[9] Let u be a fuzzy set in $F^1_{USCG}(X)$ and let u_n, $n = 1, 2, \ldots$, be fuzzy sets in $F^1_{USC}(X)$. Then $H_{\text{end}}(u_n, u) \to 0$ if and only if $H([u_n]_\alpha, [u]_\alpha) \to 0$ holds a.e. on $\alpha \in (0, 1)$, which is denoted by $H([u_n]_\alpha, [u]_\alpha) \xrightarrow{\text{a.e.}} 0$ $([0, 1])$.*

Theorem 3.18. *[9] Let U be a subset of $F^1_{USCG}(X)$. Then U is totally bounded in $(F^1_{USCG}(X), H_{\text{end}})$ if and only if $U(\alpha)$ is totally bounded in (X, d) for each $\alpha \in (0, 1]$.*

Theorem 3.19. *[9] Let U be a subset of $F^1_{USCG}(X)$. Then U is relatively compact in $(F^1_{USCG}(X), H_{\text{end}})$ if and only if $U(\alpha)$ is relatively compact in (X, d) for each $\alpha \in (0, 1]$.*

Theorem 3.20. *[9] Let U be a subset of $F^1_{USCG}(X)$. Then the following are equivalent:*

(i) *U is compact in $(F^1_{USCG}(X), H_{\text{end}})$;*

(ii) *$U(\alpha)$ is relatively compact in (X, d) for each $\alpha \in (0, 1]$ and U is closed in $(F^1_{USCG}(X), H_{\text{end}})$;*

(iii) *$U(\alpha)$ is compact in (X, d) for each $\alpha \in (0, 1]$ and U is closed in $(F^1_{USCG}(X), H_{\text{end}})$.*

Theorem 3.21. *[9] $(F^1_{USCG}(\widetilde{X}), H_{\text{end}})$ is a completion of $(F^1_{USCG}(X), H_{\text{end}})$.*

4. Characterizations of compactness in $(F^1_{USCG}(X)^p, d_p)$

In this section, we give the characterizations of total boundedness, relative compactness and compactness in $(F^1_{USCG}(X)^p, d_p)$.

The relationship between the d_p metric and the H_{end} metric given in Theorem 3.1 (see the illustrations in Remark 3.2) will be used repeatedly in the sequel.

Lemma 4.1. *If $u \in F^1_{USCG}(X)^p$, then u is p-mean left-continuous.*

19

Proof. The desired result can be proved in a similar fashion to Lemma 4.3 in [7] by replacing $\{\mathbf{0}\}$ with $\{x_0\}$, where $\mathbf{0}$ denotes the point $(\overbrace{0,\ldots,0}^{m})$ in $\mathbb{R}^m$ and x_0 is a point in X.

$\square$

Theorem 4.2. *Let U be a subset of $F_{USCG}^1(X)^p$. Then U is a relatively compact set in $(F_{USCG}^1(X)^p, d_p)$ if and only if*
(i) U is a relatively compact set in $(F_{USCG}^1(X), H_{\text{end}})$, and
(ii) U is p-mean equi-left-continuous.

Proof. Necessity. If U is a relatively compact set in $(F_{USCG}^1(X)^p, d_p)$, then by Theorem 3.1, (i) is true.

The necessity of (ii) can be proved in a similar fashion to the necessity of (ii) in Theorem 4.1 in [7], which is Theorem 5.5 in this paper.

Sufficiency. The proof is similar to the sufficiency part of Theorem 4.1 in [7]. A sketch of the proof is given as follows

Let $\{u_n\}$ be a sequence in U. To find a subsequence $\{v_n\}$ of $\{u_n\}$ which converges to $v \in F_{USCG}^1(X)^p$ according to the d_p metric, we split the proof into three steps

Step 1. Find a subsequence $\{v_n\}$ of $\{u_n\}$ and $v \in F_{USCG}^1(X)$ such that $H_{\text{end}}(v_n, v) \to 0$; that is, by Theorem 3.17,

$$H([v_n]_\alpha, [v]_\alpha) \xrightarrow{\text{a.e.}} 0 \ ([0,1]). \tag{9}$$

From (i), this step can be done immediately.

Step 2. Prove that

$$\left(\int_0^1 H([v_n]_\alpha, [v]_\alpha)^p \, d\alpha \right)^{1/p} \to 0 \text{ as } n \to +\infty. \tag{10}$$

Proceeding according to the **Step 2** in the sufficiency part of the proof of Theorem 4.1 in [7], we can obtain the desired result.

Here we mention one thing. In this proof of step 2, we need to prove the conclusion: for each $h \in (0,1]$,

$$\left(\int_h^1 H([v_n]_\alpha, [v]_\alpha)^p \, d\alpha \right)^{1/p} \to 0 \text{ as } n \to +\infty. \tag{11}$$

In the proof of the corresponding conclusion in [7], which is at Page 28 Line 6 from the bottom in [7], there is a small mistake (or misprints). In

20

the following, we give a slightly adjusted proof for the above conclusion (Obviously, the proof of the corresponding conclusion in [7] can be adjusted similarly).

Note that $[v_n]_h$ and $[v]_h$ are contained in $\overline{U(h/2)}$, which is compact in X according to Theorem 3.19. Then there is an $M(h) \geq 0$ such that

$$\max\{d(x,y) : x, y \in \overline{U(h/2)}\} \leq M(h).$$

Hence

$$H([v_n]_\alpha, [v]_\alpha) \leq M(h)$$

for $\alpha \in [h,1]$ and $n = 1, 2, \ldots$. Combined with (9) and by using the Lebesgue's dominated convergence theorem, we thus obtain (11).

Step 3. Show that $v \in F_{USCG}^1(X)^p$.

Since $v \in F_{USCG}^1(X)$, it suffices to show that $\left(\int_0^1 H([v]_\alpha, \{x_0\})^p \, d\alpha\right)^{1/p} < +\infty$ for some $x_0 \in X$, which can be proved in a similar fashion to the conclusion "$\left(\int_0^1 H([v]_\alpha, \{\mathbf{0}\})^p \, d\alpha\right)^{1/p} < +\infty$" in the **Step 3** in the sufficiency part of Theorem 4.1 in [7] by replacing $\{\mathbf{0}\}$ with $\{x_0\}$, where $x_0 \in X$. $\qquad \square$

Theorem 4.3. *Let U be a subset of $F_{USCG}^1(X)^p$. Then U is a relatively compact set in $(F_{USCG}^1(X)^p, d_p)$ if and only if*
(i) $U(\alpha)$ is relatively compact in (X,d) for each $\alpha \in (0,1]$, and
(ii) U is p-mean equi-left-continuous.

Proof. The desired result follows immediately from Theorems 3.19 and 4.2. $\qquad \square$

Theorem 4.4. *Let U be a subset of $F_{USCG}^1(X)^p$. Then U is a totally bounded set in $(F_{USCG}^1(X)^p, d_p)$ if and only if*
(i) U is a totally bounded set in $(F_{USCG}^1(X), H_{\text{end}})$, and
(ii) U is p-mean equi-left-continuous.

Proof. Necessity. Suppose that U is totally bounded in $(F_{USCG}^1(X)^p, d_p)$. Then by (2), U is a totally bounded set in $(F_{USCG}^1(X), H_{\text{end}})$; that is, (i) is true.

The necessity of (ii) can be proved in a similar fashion to the necessity of (ii) in Theorem 4.1 in [7], which is Theorem 5.5 in this paper. Let $\varepsilon > 0$. Since U is totally bounded, there exists an $\varepsilon/3$ net $\{u_1, \ldots, u_n\}$ of U. The

21

remainder proof of the p-mean equi-left-continuity of U is similar to the corresponding part of the proof of the necessity of (ii) in Theorem 4.1 in [7].

Sufficiency. Suppose that U satisfies (i) and (ii). From Theorem 3.21, U is a relatively compact set in $(F^1_{USCG}(\widetilde{X}), H_{\text{end}})$. Then, by Theorem 4.2, U is a relatively compact set in $(F^1_{USCG}(\widetilde{X})^p, d_p)$, and thus U is a totally bounded set in $(F^1_{USCG}(X)^p, d_p)$.

$\square$

Theorem 4.5. *Let U be a subset of $F^1_{USCG}(X)^p$. Then U is a totally bounded set in $(F^1_{USCG}(X)^p, d_p)$ if and only if*
(i) $U(\alpha)$ is totally bounded in (X, d) for each $\alpha \in (0, 1]$, and
(ii) U is p-mean equi-left-continuous.

Proof. The desired result follows immediately from Theorems 3.18 and 4.4.

$\square$

Theorem 4.6. *Let U be a subset of $F^1_{USCG}(X)^p$. Then U is compact in $(F^1_{USCG}(X)^p, d_p)$ if and only if*
(i) $U(\alpha)$ is relatively compact in (X, d) for each $\alpha \in (0, 1]$,
(ii) U is p-mean equi-left-continuous, and
(iii) U is a closed set in $(F^1_{USCG}(X)^p, d_p)$.

Proof.   The desired result follows immediately from Theorem 4.3.

$\square$

Theorem 4.7. *Let U be a subset of $F^1_{USCG}(X)^p$. Then U is compact in $(F^1_{USCG}(X)^p, d_p)$ if and only if*
(i) $U(\alpha)$ is compact in (X, d) for each $\alpha \in (0, 1]$,
(ii) U is p-mean equi-left-continuous, and
(iii) U is a closed set in $(F^1_{USCG}(X)^p, d_p)$.

Proof. By Theorem 4.6, to show the desired result, we only need to show that if U is compact in $(F^1_{USCG}(X)^p, d_p)$, then (i) is true.

Assume that U is compact in $(F^1_{USCG}(X)^p, d_p)$, then by (2), U is compact in $(F^1_{USCG}(X), H_{\text{end}})$, hence from Theorem 3.20, (i) is true.

$\square$

Theorem 4.8. *Let U be a subset of $F^1_{USCG}(X)^p$. Then U is a compact set in $(F^1_{USCG}(X)^p, d_p)$ if and only if*
(i) U is a compact set in $(F^1_{USCG}(X), H_{\text{end}})$, and
(ii) U is p-mean equi-left-continuous.

Proof. Necessity. Assume that U is compact in $(F^1_{USCG}(X)^p, d_p)$. Hence by Theorem 3.1, U is compact in $(F^1_{USCG}(X), H_{end})$, i.e. (i) is true. By Theorem 4.7, (ii) is true.

Sufficiency. Assume that (i) and (ii) are true. Then by Theorem 4.2, U is relatively compact in $(F^1_{USCG}(X)^p, d_p)$. To show that U is compact in $(F^1_{USCG}(X)^p, d_p)$, we only need to show that U is closed in $(F^1_{USCG}(X)^p, d_p)$.

To do this, let $\{u_n\}$ be a sequence in U and $d_p(u_n, u) \to 0$. Then by Theorem 3.1, $H_{end}(u_n, u) \to 0$. Since from (i), U is closed in $(F^1_{USCG}(X), H_{end})$, we have that $u \in U$. So U is closed in $(F^1_{USCG}(X)^p, d_p)$.

$\square$

From Theorems 4.2, 4.4 and 4.8, we obtain the following conclusion:

- Let U be a subset in $F^1_{USCG}(X)^p$. Then U is total bounded (respectively, relatively compact, compact) in $(F^1_{USCG}(X)^p, d_p)$ if and only if U is total bounded (respectively, relatively compact, compact) in $(F^1_{USCG}(X), H_{end})$ and U is p-mean equi-left-continuous.

5. Characterizations of compactness in $(F^1_{USCG}(\mathbb{R}^m)^p, d_p)$

In this section, we discuss the characterizations of total boundedness, relative compactness and compactness in $(F^1_{USCG}(\mathbb{R}^m)^p, d_p)$. We point out that the conclusions on the characterizations of total boundedness, relative compactness and compactness in $(F^1_{USCG}(\mathbb{R}^m)^p, d_p)$ given in our previous work [10] can be seen as corollaries of the corresponding results for $(F^1_{USCG}(X)^p, d_p)$ given in Section 4 of this paper. Furthermore, by using results in Sections 3 and 4, we give new characterizations of total boundedness, relative compactness and compactness in $(F^1_{USCG}(\mathbb{R}^m)^p, d_p)$.

Note that for a subset V of $\mathbb{R}^m$, the conditions (i) V is relatively compact in $\mathbb{R}^m$, (ii) V is totally bounded in $\mathbb{R}^m$, and (iii) V is bounded in $\mathbb{R}^m$, are equivalent to each other. Thus Theorems 4.3, 4.5, 4.6 and 4.7 imply the following four conclusions on the characterizations of compactness in $(F^1_{USCG}(\mathbb{R}^m)^p, d_p)$, respectively.

Corollary 5.1. *Let U be a subset of $F^1_{USCG}(\mathbb{R}^m)^p$. Then U is a relatively compact set in $(F^1_{USCG}(\mathbb{R}^m)^p, d_p)$ if and only if*
(i) $U(\alpha)$ is bounded in $\mathbb{R}^m$ for each $\alpha \in (0, 1]$, and
(ii) U is p-mean equi-left-continuous.

Corollary 5.2. *Let U be a subset of $F_{USCG}^1(\mathbb{R}^m)^p$. Then U is a totally bounded set in $(F_{USCG}^1(\mathbb{R}^m)^p, d_p)$ if and only if*
(i) $U(\alpha)$ is bounded in $\mathbb{R}^m$ for each $\alpha \in (0,1]$, and
(ii) U is p-mean equi-left-continuous.

Corollary 5.3. *Let U be a subset of $F_{USCG}^1(\mathbb{R}^m)^p$. Then U is compact in $(F_{USCG}^1(\mathbb{R}^m)^p, d_p)$ if and only if*
(i) $U(\alpha)$ is bounded in $\mathbb{R}^m$ for each $\alpha \in (0,1]$,
(ii) U is p-mean equi-left-continuous, and
(iii) U is a closed set in $(F_{USCG}^1(\mathbb{R}^m)^p, d_p)$.

Corollary 5.4. *Let U be a subset of $F_{USCG}^1(\mathbb{R}^m)^p$. Then U is compact in $(F_{USCG}^1(\mathbb{R}^m)^p, d_p)$ if and only if*
(i) $U(\alpha)$ is compact in $\mathbb{R}^m$ for each $\alpha \in (0,1]$,
(ii) U is p-mean equi-left-continuous, and
(iii) U is a closed set in $(F_{USCG}^1(\mathbb{R}^m)^p, d_p)$.

In [7], we have obtained the following three conclusions on the characterizations of compactness in $(F_{USCG}^1(\mathbb{R}^m)^p, d_p)$.

Theorem 5.5. (Theorem 4.1 in [7]) *Let U be a subset of $F_{USCG}^1(\mathbb{R}^m)^p$. Then U is a relatively compact set in $(F_{USCG}^1(\mathbb{R}^m)^p, d_p)$ if and only if*
(i) U is uniformly p-mean bounded, and
(ii) U is p-mean equi-left-continuous.

Theorem 5.6. (Theorem 4.2 in [7]) *Let U be a subset of $F_{USCG}^1(\mathbb{R}^m)^p$. Then U is a totally bounded set in $(F_{USCG}^1(\mathbb{R}^m)^p, d_p)$ if and only if*
(i) U is uniformly p-mean bounded, and
(ii) U is p-mean equi-left-continuous.

Theorem 5.7. (Theorem 4.3 in [7]) *Let U be a subset of $F_{USCG}^1(\mathbb{R}^m)^p$. Then U is compact in $(F_{USCG}^1(\mathbb{R}^m)^p, d_p)$ if and only if*
(i) U is uniformly p-mean bounded,
(ii) U is p-mean equi-left-continuous, and
(iii) U is a closed set in $(F_{USCG}^1(\mathbb{R}^m)^p, d_p)$.

Remark 5.8. From Theorem 3.14, we can see:
Corollary 5.1 implies Theorem 4.1 in [7] (which is Theorem 5.5 in this paper), and the converse is true;
Corollary 5.2 implies Theorem 4.2 in [7] (which is Theorem 5.6 in this paper),

and the converse is true;

Corollary 5.3 implies Theorem 4.3 in [7] (which is Theorem 5.7 in this paper), and the converse is true.

So the characterizations of compactness for $(F^1_{USCG}(\mathbb{R}^m)^p, d_p)$ in [7] (Theorems 4.1, 4.2 and 4.3 in [7]) can be seen as corollaries of the characterizations of compactness for $(F^1_{USCG}(X)^p, d_p)$ in this paper (Theorems 4.3, 4.5 and 4.6). The results of the characterizations of compactness for $(F^1_{USCG}(X)^p, d_p)$ in this paper generalize the corresponding results for $(F^1_{USCG}(\mathbb{R}^m)^p, d_p)$ in [7].

In the following, we give new characterizations of compactness for $(F^1_{USCG}(\mathbb{R}^m)^p, d_p)$.

Using Theorem 3.14, we can obtain the following characterizations of compactness for $(F^1_{USCG}(\mathbb{R}^m)^p, d_p)$ from Corollaries 5.1, 5.2 and 5.3.

Theorem 5.9. *Let U be a subset of $F^1_{USCG}(\mathbb{R}^m)^p$. Then U is a relatively compact set in $(F^1_{USCG}(\mathbb{R}^m)^p, d_p)$ if and only if*
(i) There exists an $h \in (0, 1)$ such that $U(h)$ is bounded in $\mathbb{R}^m$, and
(ii) U is p-mean equi-left-continuous.

Theorem 5.10. *Let U be a subset of $F^1_{USCG}(\mathbb{R}^m)^p$. Then U is a totally bounded set in $(F^1_{USCG}(\mathbb{R}^m)^p, d_p)$ if and only if*
(i) There exists an $h \in (0, 1)$ such that $U(h)$ is bounded in $\mathbb{R}^m$, and
(ii) U is p-mean equi-left-continuous.

Theorem 5.11. *Let U be a subset of $F^1_{USCG}(\mathbb{R}^m)^p$. Then U is compact in $(F^1_{USCG}(\mathbb{R}^m)^p, d_p)$ if and only if*
(i) There exists an $h \in (0, 1)$ such that $U(h)$ is bounded in $\mathbb{R}^m$,
(ii) U is p-mean equi-left-continuous, and
(iii) U is a closed set in $(F^1_{USCG}(\mathbb{R}^m)^p, d_p)$.

6. Completion of $(F^1_{USCG}(X)^p, d_p)$

In this section, we show that $(F^1_{USCG}(\widetilde{X})^p, d_p)$ is a completion of $(F^1_{USCB}(X), d_p)$, and thus a completion of $(F^1_{USCG}(X)^p, d_p)$.

Theorem 6.1. *(X, d) is complete if and only if $(F^1_{USCG}(X)^p, d_p)$ is complete.*

Proof. Necessity. Suppose that (X, d) is complete. To show that $(F^1_{USCG}(X)^p, d_p)$ is complete, we only need to show that each Cauchy sequence in $(F^1_{USCG}(X)^p, d_p)$ is relatively compact.

Let $\{u_n : n \in \mathbb{N}\}$ be a Cauchy sequence in $(F^1_{USCG}(X)^p, d_p)$. Then $\{u_n : n \in \mathbb{N}\}$ is totally bounded in $(F^1_{USCG}(X)^p, d_p)$. By Theorems 4.2 and 4.4, to show that $\{u_n : n \in \mathbb{N}\}$ is relatively compact in $(F^1_{USCG}(X)^p, d_p)$, we only need to show that $\{u_n : n \in \mathbb{N}\}$ is relatively compact in $(F^1_{USCG}(X)^p, H_{\text{end}})$.

By Theorem 4.4, $\{u_n : n \in \mathbb{N}\}$ is totally bounded in $(F^1_{USCG}(X), H_{\text{end}})$. Since, by Theorem 6.1 in [9], $(F^1_{USCG}(X), H_{\text{end}})$ is complete, and thus $\{u_n : n \in \mathbb{N}\}$ is relatively compact in $(F^1_{USCG}(X), H_{\text{end}})$.

Sufficiency. Suppose that $(F^1_{USCG}(X)^p, d_p)$ is complete. Let $\widehat{X} = \{\widehat{x} : x \in X\}$. Then $\widehat{X} \subseteq F^1_{USCB}(X)$. Define $f : X \to \widehat{X}$ by $f(x) = \widehat{x}$. Note that $d(x, y) = d_p(\widehat{x}, \widehat{y})$. Hence f is a isometry from X to $\widehat{X}$. If $\{\widehat{x_n}\}$ converges to $u \in F^1_{USCG}(X)^p$, then there exists an $x \in X$ such that $[u]_\alpha = \{x\}$ for all $\alpha \in [0, 1]$; that is $u = \widehat{x}$. Thus $\widehat{X}$ is a closed subspace of $(F^1_{USCG}(X)^p, d_p)$. So (X, d) is isometric to a closed subspace of $(F^1_{USCG}(X)^p, d_p)$, and then (X, d) is complete.

$\square$

Corollary 6.2. $(F^1_{USCG}(\mathbb{R}^m)^p, d_p)$ *is complete.*

Proof. Since $\mathbb{R}^m$ is complete, the desired result follows immediately from Theorem 6.1.

$\square$

Remark 6.3. Corollary 6.2 is Theorem 5.1 in [7]. So Theorem 6.1 in this paper generalizes Theorem 5.1 in [7].

For $u \in F^1_{USCG}(X)$ and $\varepsilon > 0$, define $u^\varepsilon \in F^1_{USCB}(X)$ by

$$[u^\varepsilon]_\alpha = \begin{cases} [u]_\alpha, & \alpha \in (\varepsilon, 1], \\ [u]_\varepsilon, & \alpha \in [0, \varepsilon]. \end{cases}$$

Theorem 6.4. $F^1_{USCB}(X)$ *is a dense set in* $(F^1_{USCG}(X)^p, d_p)$.

Proof. The desired result can be proved in a similar fashion to Theorem 5.2 in [7]. In fact, it is shown that for each $v \in F^1_{USCG}(X)^p$, $d_p(v^{(1/n)}, v) \to 0$.

$\square$

Theorem 6.5. $(F^1_{USCG}(\widetilde{X})^p, d_p)$ *is a completion of* $(F^1_{USCB}(X), d_p)$.

Proof. In the proof of Theorem 6.3 in [9], we show the following conclusion

26

- for each $v \in F_{USCB}^1(\widetilde{X})$ and each $\varepsilon > 0$, there is a $w \in F_{USCB}^1(X)$ such that $H([w]_\alpha, [v]_\alpha) \leq \varepsilon$ for all $\alpha \in [0,1]$.

Thus we have $d_p(v, w) \leq \varepsilon$. This means that $F_{USCB}^1(X)$ is dense in $(F_{USCB}^1(\widetilde{X}), d_p)$.

From Theorem 6.4, we know that $F_{USCB}^1(\widetilde{X})$ is dense in $(F_{USCG}^1(\widetilde{X})^p, d_p)$.

Combined the above conclusions, we obtain that $F_{USCB}^1(X)$ is dense in $(F_{USCG}^1(\widetilde{X})^p, d_p)$. By Theorem 6.1, $(F_{USCG}^1(\widetilde{X})^p, d_p)$ is complete. So $(F_{USCG}^1(\widetilde{X})^p, d_p)$ is a completion of $(F_{USCB}^1(X), d_p)$.

$\square$

Corollary 6.6. $(F_{USCG}^1(\widetilde{X})^p, d_p)$ is a completion of $(F_{USCG}^1(X)^p, d_p)$.

Proof. Since $F_{USCB}^1(X) \subseteq F_{USCG}^1(X)^p \subseteq F_{USCG}^1(\widetilde{X})^p$, the desired result follows immediately from Theorem 6.5.

$\square$

7. Measurability of function $H([u]_\alpha, [v]_\alpha)$

For $u, v \in F_{USC}^1(X)$, $d_p(u, v)$ is well-defined if and only if $H([u]_\alpha, [v]_\alpha)$ is a measurable function of α on $[0,1]$. So it is important to discuss the measurability of the function $H([u]_\alpha, [v]_\alpha)$ of α on $[0,1]$. In this section, we give some fundamental conclusions on this topic.

We uniformly use H to denote the Hausdorff metric on $C(X)$ induced by d_X, where (X, d_X) is a certain metric space. The meaning of H can be judged according to the context.

We have pointed out the following statements on the measurability of the function $H([u]_\alpha, [v]_\alpha)$ (See [8] or [10] which was submitted on 2019.07.06).

- For $u \in F_{USC}^1(X)$ and $x_0 \in X$, $H([u]_\alpha, \{x_0\})$ is a measurable function of α on $[0,1]$.

- For $u, v \in F_{USC}^1(\mathbb{R}^m)$, $H([u]_\alpha, [v]_\alpha)$ is a measurable function of α on $[0,1]$.

- For $u, v \in F_{USCG}^1(X)$, $H([u]_\alpha, [v]_\alpha)$ is a measurable function of α on $[0,1]$.

- There exists a metric space X and $u, v \in F_{USC}^1(X)$ such that $H([u]_\alpha, [v]_\alpha)$ is a non-measurable function of α on $[0,1]$.

27

In this section, we first give the proofs of the first three statements and the example to show the last statement (we submitted the proofs of the first three statements in [11]). Then we give some improvements of these statements.

Proposition 7.1. *For $u \in F^1_{USC}(X)$ and $x_0 \in X$, $H([u]_\alpha, \{x_0\})$ is a measurable function of α on $[0, 1]$.*

Proof. We can see that for $0 \le \alpha \le \beta \le 1$,

$$H([u]_\alpha, \{x_0\}) = \sup_{x \in [u]_\alpha} d(x, x_0) \ge \sup_{x \in [u]_\beta} d(x, x_0) = H([u]_\beta, \{x_0\}).$$

The desired result follows from the fact that $H([u]_\alpha, \{x_0\})$ is a monotone function of α on $[0, 1]$.

$\square$

Let $\widehat{\mathbb{R}} = \mathbb{R} \cup \{-\infty\} \cup \{+\infty\}$. Let $[\mu, \nu]$ be an interval in $\mathbb{R}$, $\alpha \in (\mu, \nu]$, f a function from $[\mu, \nu]$ to $\widehat{\mathbb{R}}$. **$\liminf_{\gamma \to \alpha-} f(\gamma)$** is defined by

$$\liminf_{\gamma \to \alpha-} f(\gamma) := \inf\{x \in \widehat{\mathbb{R}} : \text{there is a sequence } \{\gamma_n\} \text{ in } [\mu, \nu]$$

$$\text{such that } \gamma_n \to \alpha- \text{ and } x = \lim_{n \to +\infty} f(\gamma_n)\},$$

here $\lim_{n \to +\infty} f(\gamma_n) = +\infty$ and $\lim_{n \to +\infty} f(\gamma_n) = -\infty$ are possible.

For $f : [\mu, \nu] \to \widehat{\mathbb{R}}$ and $\alpha \in (\mu, \nu]$, $\liminf_{\gamma \to \alpha-} f(\gamma)$ exists. $\liminf_{\gamma \to \alpha-} f(\gamma) = +\infty$ and $\liminf_{\gamma \to \alpha-} f(\gamma) = -\infty$ are possible.

We can check that $\liminf_{\gamma \to \alpha-} f(\gamma) = \min\{x \in \widehat{\mathbb{R}} : \text{there is a sequence } \{\gamma_n\} \text{ such that } \gamma_n \to \alpha- \text{ and } x = \lim_{n \to +\infty} f(\gamma_n)\}$. Clearly if $\lim_{\gamma \to \alpha-} f(\gamma)$ exists, then $\lim_{\gamma \to \alpha-} f(\gamma) = \liminf_{\gamma \to \alpha-} f(\gamma)$.

Let f be a function from $[\mu, \nu]$ to $\widehat{\mathbb{R}}$ and $r \in \mathbb{R}$. The symbol $\{f > r\}$ is used to denote the set $\{\alpha \in [\mu, \nu] : f(\alpha) > r\}$.

Lemma 7.2. *Let $[\mu, \nu]$ be an interval, f a function from $[\mu, \nu]$ to $\widehat{\mathbb{R}}$, and $\alpha \in (\mu, \nu]$. Then the following properties are equivalent:*
(i) For each $r \in \mathbb{R}$, if $\alpha \in \{f > r\}$, then there exists $\delta(\alpha) > 0$ such that $[\alpha - \delta(\alpha), \alpha] \subseteq \{f > r\}$;
(ii) $f(\alpha) \le \liminf_{\gamma \to \alpha-} f(\gamma)$.

Proof. The equivalence of properties (i) and (ii) follows from basic analysis.

Assume that (i) is true. If $f(\alpha) = -\infty$, then (ii) is true. Now suppose that $f(\alpha) > -\infty$. Given $r \in \mathbb{R}$ with $f(\alpha) > r$. Then there exists $\delta(\alpha) > 0$ such that $[\alpha - \delta(\alpha), \alpha] \subseteq \{f > r\}$, thus $\liminf_{\gamma \to \alpha-} f(\gamma) \geq r$. From the arbitrariness of $r \in (-\infty, f(\alpha))$, we have $f(\alpha) \leq \liminf_{\gamma \to \alpha-} f(\gamma)$. So (ii) is true.

Assume that (ii) is true. Let $r \in \mathbb{R}$. If $\alpha \in \{f > r\}$, then $\liminf_{\gamma \to \alpha-} f(\gamma) > r$. We claim that there exists $\delta(\alpha) > 0$ such that $[\alpha - \delta(\alpha), \alpha] \subseteq \{f > r\}$. Otherwise there exists $\{\alpha_n\}$ in $[\mu, \nu]$ such that $\alpha_n \to \alpha-$ and $f(\alpha_n) \leq r$, which contradicts $\liminf_{\gamma \to \alpha-} f(\gamma) > r$.

$\square$

Proposition 7.3. *Let $[\mu, \nu]$ be an interval and let f be a function from $[\mu, \nu]$ to $\widehat{\mathbb{R}}$. If $f(\alpha) \leq \liminf_{\gamma \to \alpha-} f(\gamma)$ for all $\alpha \in (\mu, \nu]$, then f is a measurable function on $[\mu, \nu]$.*

Proof. We only need to show that for each $r \in \mathbb{R}$, the set $\{f > r\}$ is a measurable set.

Step (i) For each $r \in \mathbb{R}$, if $\alpha > \mu$ and $\alpha \in \{f > r\}$, then there exists $\delta(\alpha) > 0$ such that $[\alpha - \delta(\alpha), \alpha] \subseteq \{f > r\}$.

From Lemma 7.2, this conclusion is equivalent to the conclusion that $f(\alpha) \leq \liminf_{\gamma \to \alpha-} f(\gamma)$ for all $\alpha \in (\mu, \nu]$.

Step (ii) For each $r \in \mathbb{R}$, if $\{f > r\} \setminus \{\mu\} \neq \emptyset$, then $\{f > r\} \setminus \{\mu\}$ is a union of disjoint positive length intervals.

Suppose that $\{f > r\} \setminus \{\mu\} \neq \emptyset$. For $x \in \{f > r\} \setminus \{\mu\}$, let $\overset{\frown}{x} = \bigcup\{[a, b] : x \in [a, b] \subseteq \{f > r\} \setminus \{\mu\}\}$, i.e. $\overset{\frown}{x}$ is the largest interval in $\{f > r\} \setminus \{\mu\}$ which contains x. Then by step (i), $\overset{\frown}{x}$ is a positive length interval. Note that for $x, y \in \{f > r\} \setminus \{\mu\}$, if $\overset{\frown}{x} \cap \overset{\frown}{y} \neq \emptyset$, then $\overset{\frown}{x} = \overset{\frown}{y}$. Thus $\{f > r\} \setminus \{\mu\}$ is a union of disjoint positive length intervals.

Step (iii) For each $r \in \mathbb{R}$, $\{f > r\}$ is a measurable set.

Clearly, if positive length intervals are disjoint, then these positive length intervals are at most countable. Thus, from step (ii), $\{f > r\} \setminus \{\mu\}$ is a measurable set. So $\{f > r\}$ is a measurable set.

$\square$

Corollary 7.4. *Let $[\mu, \nu]$ be an interval in $\mathbb{R}$ and let f be a function from $[\mu, \nu]$ to $\widehat{\mathbb{R}}$. If f is left-continuous at $\alpha \in (\mu, \nu]$, i.e., $f(\alpha) = \lim_{\gamma \to \alpha-} f(\gamma)$, then f is a measurable function on $[\mu, \nu]$.*

Proof. f is left-continuous on $(\mu, \nu]$ means that $f(\alpha) = \lim_{\gamma \to \alpha-} f(\gamma) = \liminf_{\gamma \to \alpha-} f(\gamma)$ for all $\alpha \in (\mu, \nu]$. So the desired result follows immediately from Proposition 7.3.

$\square$

For $u, v \in F_{USC}^1(X)$ and $r \in \mathbb{R}$, we use the symbol $\{H(u, v) > r\}$ to denote the set $\{\alpha \in [0, 1] : H([u]_\alpha, [v]_\alpha) > r\}$.

Proposition 7.5. *For $u, v \in F_{USC}^1(\mathbb{R}^m)$, $H([u]_\alpha, [v]_\alpha)$ is a measurable function of α on $[0, 1]$.*

Proof. From Proposition 7.3, to show the desired results, we only need to show the conclusion that $H([u]_\alpha, [v]_\alpha) \leq \liminf_{\gamma \to \alpha-} H([u]_\gamma, [v]_\gamma)$ for all $\alpha \in (0, 1]$, which, by Lemma 7.2, is equivalent to the following conclusion

- For each $r \in \mathbb{R}$, if $\alpha > 0$ and $\alpha \in \{H(u, v) > r\}$, then there exists $\delta(\alpha) > 0$ such that $[\alpha - \delta(\alpha), \alpha] \subseteq \{H(u, v) > r\}$.

We proceed by contradiction. Assume that there is a $r \in \mathbb{R}$ and $\alpha > 0$ with $\alpha \in \{H(u, v) > r\}$ such that for each $\delta > 0$, $[\alpha - \delta, \alpha] \nsubseteq \{H(u, v) > r\}$. Then there exists a sequence $\{\gamma_n\}$ in $[0, \alpha)$ such that for $n = 1, 2, \ldots$, $\gamma_{n+1} > \gamma_n$, $\gamma_n \to \alpha$, and
$$H([u]_{\gamma_n}, [v]_{\gamma_n}) \leq r.$$

Given $x \in [u]_\alpha$, then $d(x, [v]_{\gamma_n}) \leq H([u]_{\gamma_n}, [v]_{\gamma_n}) \leq r$. Therefore there exist $y_n \in [v]_{\gamma_n}$ such that $d(x, y_n) = d(x, [v]_{\gamma_n}) \leq r$. Hence there is a subsequence $\{y_{n_i}\}$ of $\{y_n\}$ such that $\{y_{n_i}\}$ converges to $y \in \mathbb{R}^m$. Note that $d(x, y) \leq r$ and $y \in \cap [v]_{\gamma_{n_i}} = [v]_\alpha$, so we have $d(x, [v]_\alpha) \leq r$.

From the arbitrariness of x, $H^*([u]_\alpha, [v]_\alpha) \leq r$. Similarly, we can deduce that $H^*([v]_\alpha, [u]_\alpha) \leq r$. Thus $H([u]_\alpha, [v]_\alpha) \leq r$, which is a contradiction.

$\square$

The following Proposition 7.6 is Lemma 4.4 in [8].

Proposition 7.6. *[8] Let $U_n \in K(X)$ for $n = 1, 2, \ldots$.*
(i) If $U_1 \supseteq U_2 \supseteq \ldots \supseteq U_n \supseteq \ldots$, then $U = \bigcap_{n=1}^{+\infty} U_n \in K(X)$ and $H(U_n, U) \to 0$ as $n \to +\infty$.
(ii) If $V_1 \subseteq V_2 \subseteq \ldots \subseteq V_n \subseteq \ldots$ and $V = \overline{\bigcup_{n=1}^{+\infty} V_n} \in K(X)$, then $H(V_n, V) \to 0$ as $n \to +\infty$.

Proof. (i) is easy to show. Since for $n = 1, 2, \ldots$, U_n is closed, then U is a closed subset of the compact set U_1. Hence U is compact. Suppose that $H(U_n, U) \nrightarrow 0$. Then there is an $\varepsilon_0 > 0$ such that $H(U_n, U) > \varepsilon_0$ for $n = 1, 2, \ldots$. Hence for each $n = 1, 2, \ldots$ there exists $x_n \in U_n$ such that

$$d(x_n, U) > \varepsilon_0. \tag{12}$$

Since U_1 is compact, then there is a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that x_{n_i} converges to $x \in U_1$. Note that $x \in \bigcap_{i=1}^{+\infty} U_{n_i} = U$, which contradicts (12).

To prove (ii). Suppose that $H(V_n, V) \nrightarrow 0$. Then there is an $\varepsilon_0 > 0$ such that $H(V_n, V) > \varepsilon_0$ for $n = 1, 2, \ldots$. Hence for each $n = 1, 2, \ldots$ there exists $x_n \in V$ such that

$$d(x_n, V_n) > \varepsilon_0. \tag{13}$$

Since V is compact, there is a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that x_{n_i} converges to $x \in V$. From the definition of V, there exists $\{y_n\}$ such that $y_n \in V_n$ and $y_n \to x$. Thus $d(x, V_n) \to 0$, which contradicts (13). $\qquad \square$

Proposition 7.7. *(i) For $u, v \in F_{USCG}^1(X)$, $H([u]_\alpha, [v]_\alpha)$ is left-continuous at $\alpha \in (0, 1]$;*
(ii) For $u, v \in F_{USCB}^1(X)$, $H([u]_\alpha, [v]_\alpha)$ is right-continuous at $\alpha = 0$.

Proof. To show (i), let $u, v \in F_{USCG}^1(X)$. Note that $H([u]_\alpha, [v]_\alpha)$ is finite at $\alpha \in (0, 1]$ and for $\alpha, \beta \in (0, 1]$,

$$|H([u]_\alpha, [v]_\alpha) - H([u]_\beta, [v]_\beta)| \leq H([u]_\alpha, [u]_\beta) + H([v]_\alpha, [v]_\beta).$$

By Proposition 7.6 (i), $\lim_{\beta \to \alpha-}(H([u]_\alpha, [u]_\beta) + H([v]_\alpha, [v]_\beta)) = 0$, and therefore for each $\alpha \in (0, 1]$,

$$\lim_{\beta \to \alpha-} H([u]_\beta, [v]_\beta) = H([u]_\alpha, [v]_\alpha),$$

i.e. $H([u]_\alpha, [v]_\alpha)$ is left-continuous at $\alpha \in (0, 1]$.

The proof of (ii) is similar to that of (i). Note that for $u, v \in F_{USCB}^1(X)$ and $\alpha \in (0, 1]$,

$$|H([u]_\alpha, [v]_\alpha) - H([u]_0, [v]_0)| \leq H([u]_\alpha, [u]_0) + H([v]_\alpha, [v]_0).$$

By Proposition 7.6 (ii), $\lim_{\alpha \to 0+}(H([u]_\alpha, [u]_0) + H([v]_\alpha, [v]_0)) = 0$, and therefore $\lim_{\alpha \to 0+} H([u]_\alpha, [v]_\alpha) = H([u]_0, [v]_0)$, i.e. $H([u]_\alpha, [v]_\alpha)$ is right-continuous at $\alpha = 0$. $\qquad \square$

31

Proposition 7.8. *For $u, v \in F^1_{USCG}(X)$, $H([u]_\alpha, [v]_\alpha)$ is a measurable function of α on $[0,1]$.*

Proof. The desired result follows immediately from Corollary 7.4 and Proposition 7.7 (i).

<div style="text-align: right">□</div>

Proposition 7.9. *For $u \in F^1_{USCG}(X)$, $h \in [0,1)$,*
(i) $H([u]_\alpha, [u]_{\alpha-h})$ is left-continuous at $\alpha \in (h,1]$;
(ii) $H([u]_\alpha, [u]_{\alpha-h})$ is a measurable function of α on $[h,1]$;
(iii) $H([u]_\alpha, [u]_{\alpha+h})$ is left-continuous at $\alpha \in (0, 1-h]$;
(iv) $H([u]_\alpha, [u]_{\alpha+h})$ is a measurable function of α on $[0, 1-h]$.

Proof. The proofs of (i) and (iii) are similar to that of Proposition 7.7 (i). The proofs of (ii) and (iv) are similar to that of Proposition 7.8.

Note that $H([u]_\alpha, [u]_{\alpha-h})$ is finite at $\alpha \in (h,1]$ and for $\alpha, \beta \in (h,1]$

$$|H([u]_\alpha, [u]_{\alpha-h}) - H([u]_\beta, [u]_{\beta-h})| \le H([u]_\alpha, [u]_\beta) + H([u]_{\alpha-h}, [u]_{\beta-h}).$$

By Proposition 7.6 (i), $\lim_{\beta \to \alpha-}(H([u]_\alpha, [u]_\beta) + H([u]_{\alpha-h}, [u]_{\beta-h})) = 0$, and therefore for each $\alpha \in (h,1]$,

$$\lim_{\beta \to \alpha-} H([u]_\beta, [u]_{\beta-h}) = H([u]_\alpha, [u]_{\alpha-h}),$$

i.e. $H([u]_\alpha, [u]_{\alpha-h})$ is left-continuous at $\alpha \in (h,1]$. So (i) is true, and thus from Corollary 7.4, (ii) is true.

Note that $H([u]_\alpha, [u]_{\alpha+h})$ is finite at $\alpha \in (0, 1-h]$ and for $\alpha, \beta \in (0, 1-h]$

$$|H([u]_\alpha, [u]_{\alpha+h}) - H([u]_\beta, [u]_{\beta+h})| \le H([u]_\alpha, [u]_\beta) + H([u]_{\alpha+h}, [u]_{\beta+h}).$$

By Proposition 7.6 (i), $\lim_{\beta \to \alpha-}(H([u]_\alpha, [u]_\beta) + H([u]_{\alpha+h}, [u]_{\beta+h})) = 0$, and therefore for each $\alpha \in (0, 1-h]$,

$$\lim_{\beta \to \alpha-} H([u]_\beta, [u]_{\beta+h}) = H([u]_\alpha, [u]_{\alpha+h}),$$

i.e. $H([u]_\alpha, [u]_{\alpha+h})$ is left-continuous at $\alpha \in (0, 1-h]$. So (iii) is true, and thus from Corollary 7.4, (iv) is true.

<div style="text-align: right">□</div>

To give the example which shows the last statement presented in [8] which is listed at the beginning of this section, we need some conclusions at first.

The following representation theorem should be a known conclusion.

Theorem 7.10. *Let X be a set. Given $u \in F(X)$, then for all $\alpha \in (0,1]$, $[u]_\alpha = \cap_{\beta<\alpha}[u]_\beta$.*

Conversely, suppose that $\{u(\alpha) : \alpha \in (0,1]\}$ is a family of sets in X satisfying $u(\alpha) = \cap_{\beta<\alpha}u(\beta)$ for all $\alpha \in (0,1]$. Define $v \in F(X)$ by $v(x) := \sup\{\alpha : x \in u(\alpha)\}$ ($\sup \emptyset = 0$). Then $[v]_\alpha = u(\alpha)$ for all $\alpha \in (0,1]$.

Proof. (The proof is routine.) Let $u \in F(X)$ and $\alpha \in (0,1]$. For each $x \in X$, $x \in [u]_\alpha \Leftrightarrow u(x) \geq \alpha \Leftrightarrow$ for each $\beta < \alpha$, $u(x) \geq \beta \Leftrightarrow$ for each $\beta < \alpha$, $x \in [u]_\beta$. So $[u]_\alpha = \cap_{\beta<\alpha}[u]_\beta$.

Conversely, suppose that $\{u(\alpha) : \alpha \in (0,1]\}$ is a family of sets in X satisfying $u(\alpha) = \cap_{\beta<\alpha}u(\beta)$ for all $\alpha \in (0,1]$. Define $v \in F(X)$ by $v(x) := \sup\{\alpha : x \in u(\alpha)\}$ ($\sup \emptyset = 0$). To show $[v]_\alpha = u(\alpha)$ for all $\alpha \in (0,1]$, it suffices to show that for each $\alpha \in (0,1]$, $[v]_\alpha \supseteq u(\alpha)$ and $[v]_\alpha \subseteq u(\alpha)$.

Let $\alpha \in (0,1]$. For each $x \in X$, if $x \in u(\alpha)$, then clearly $v(x) \geq \alpha$, i.e. $x \in [v]_\alpha$. So $[v]_\alpha \supseteq u(\alpha)$.

For each $x \in X$, if $x \in [v]_\alpha$, i.e. $v(x) \geq \alpha$. Then $\sup\{\beta : x \in u(\beta)\} \geq \alpha$, hence for $n = 1, 2, \ldots$, there exists β_n such that $1 \geq \beta_n \geq \alpha - 1/n$ and $x \in u(\beta_n)$. Set $\gamma = \sup_{n=1}^{+\infty}\beta_n$. Then $1 \geq \gamma \geq \alpha$ and thus $x \in \cap_{n=1}^{+\infty}u(\beta_n) = u(\gamma) \subseteq u(\alpha)$. So $[v]_\alpha \subseteq u(\alpha)$.

$\square$

Let (Y, ρ) be an extended metric space. For $y \in Y$ and $\varepsilon > 0$, let $B(y, \varepsilon)$ denote the set $\{z \in Y : \rho(y, z) < \varepsilon\}$. $\{B(y, \varepsilon) : y \in Y, \varepsilon > 0\}$ is a basis for the topology induced by ρ on Y. The closure of a set A in (Y, ρ), denoted by $\overline{A}$, refers to the closure of A in Y according to the topology induced by ρ on Y. Then $x \in \overline{A}$ if and only if there is a sequence $\{x_n\}$ in Y such that $\rho(x_n, x) \to 0$. So $x \in \overline{A}$ if and only if $\rho(x, A) = 0$.

Here we mention that if (Y, ρ) is an extended metric space, then the Hausdorff distance H on $C(Y)$ induced by ρ using (1) is an extended metric on $C(Y)$, where $C(Y)$ denotes the set of nonempty closed sets in (Y, ρ). It can be seen that H satisfies positivity and symmetry. To show that H satisfies the triangle inequality, we only need to show that

$$H^*(U, W) \leq H^*(U, V) + H^*(V, W) \tag{14}$$

for $U, V, W \in C(Y)$. To do this, let $x \in U$. Then

$$\rho(x, W) \leq \inf_{y \in V} \inf_{z \in W} \{\rho(x, y) + \rho(y, z)\}$$
$$\leq \inf_{y \in V} \{\rho(x, y) + \rho(y, W)\}$$

33

$$\leq \inf_{y \in V} \rho(x, y) + H^*(V, W)$$
$$= \rho(x, V) + H^*(V, W)$$
$$\leq H^*(U, V) + H^*(V, W).$$

From the arbitrariness of x in U, we obtain (14). So the Hausdorff distance H on $C(Y)$ is the Hausdorff extended metric.

For simplicity, we refer to both the Hausdorff extended metric and the Hausdorff metric as the Hausdorff metric in this paper.

For an extended metric space (Y, ρ), we define

$$F_{USC}(Y) = \{u \in F(Y) : [u]_\alpha \text{ is closed in } (Y, \rho) \text{ for } \alpha \in (0, 1]\}.$$

Let Γ be a set, and for each $\gamma \in \Gamma$, let (X_γ, d_γ) be a metric space. Define an extended metric d on $\prod_{\gamma \in \Gamma} X_\gamma$ as

$$d(x, y) := \sup\{d_\gamma(x_\gamma, y_\gamma) : \gamma \in \Gamma\} \tag{15}$$

for $x = (x_\gamma)_{\gamma \in \Gamma}$ and $y = (y_\gamma)_{\gamma \in \Gamma}$.

We use the symbol $\prod_{\gamma \in \Gamma}(X_\gamma, d_\gamma)$ to denote the extended metric space $(\prod_{\gamma \in \Gamma} X_\gamma, d)$. If not mentioned specially, we suppose by default that the extended metric on $\prod_{\gamma \in \Gamma} X_\gamma$ is the d given by (15).

Let $u_\gamma \in F(X_\gamma)$, $\gamma \in \Gamma$. Define $u \in F(\prod_{\gamma \in \Gamma} X_\gamma)$ as

$$[u]_\alpha = \prod_{\gamma \in \Gamma} [u_\gamma]_\alpha \text{ for each } \alpha \in (0, 1]. \tag{16}$$

We use $\prod_{\gamma \in \Gamma} u_\gamma$ to denote the fuzzy set u given by (16).

From Theorem 7.10, u is well-defined because for each $\alpha \in (0, 1]$,

$$[u]_\alpha = \prod_{\gamma \in \Gamma} [u_\gamma]_\alpha = \bigcap_{\beta < \alpha} \prod_{\gamma \in \Gamma} [u_\gamma]_\beta = \bigcap_{\beta < \alpha} [u]_\beta.$$

In this paper, if not mentioned specially, we use $\overline{S}$ to denote the closure of S in a certain extended metric space (X, d_X). For a set $S \subseteq X_\gamma$, $\gamma \in \Gamma$, we use $\overline{S}$ to denote the closure of S in (X_γ, d_γ). For a set $S \subseteq \prod_{\gamma \in \Gamma} X_\gamma$, we also use $\overline{S}$ to denote the closure of S in $(\prod_{\gamma \in \Gamma} X_\gamma, d)$. The readers can judge the meaning of $\overline{S}$ according to the context.

Lemma 7.11. *Let Γ be a set, and for each $\gamma \in \Gamma$, let (X_γ, d_γ) be a metric space. If $A_\gamma \subseteq X_\gamma$ for $\gamma \in \Gamma$, then $\overline{\prod_{\gamma \in \Gamma} A_\gamma} = \prod_{\gamma \in \Gamma} \overline{A_\gamma}$.*

Proof. Clearly $\overline{\prod_{\gamma \in \Gamma} A_\gamma} \subseteq \prod_{\gamma \in \Gamma} \overline{A_\gamma}$.

Conversely, if $x = (x_\gamma)_{\gamma \in \Gamma} \in \prod_{\gamma \in \Gamma} \overline{A_\gamma}$, then for each $\varepsilon > 0$, there exists $y = (y_\gamma)_{\gamma \in \Gamma} \in \prod_{\gamma \in \Gamma} A_\gamma$ such that $d_\gamma(x_\gamma, y_\gamma) \leq \varepsilon$ for all $\gamma \in \Gamma$. So $d(x, y) \leq \varepsilon$. From the arbitrariness of $\varepsilon > 0$, we have $x \in \overline{\prod_{\gamma \in \Gamma} A_\gamma}$. Thus $\overline{\prod_{\gamma \in \Gamma} A_\gamma} \supseteq \prod_{\gamma \in \Gamma} \overline{A_\gamma}$.

In summary, $\overline{\prod_{\gamma \in \Gamma} A_\gamma} = \prod_{\gamma \in \Gamma} \overline{A_\gamma}$.

$\square$

Theorem 7.12. *Let Γ be a set, and for each $\gamma \in \Gamma$, let (X_γ, d_γ) be a metric space. If $u_\gamma \in F_{USC}(X_\gamma)$ for each $\gamma \in \Gamma$, then $u = \prod_{\gamma \in \Gamma} u_\gamma$ is a fuzzy set in $F_{USC}(\prod_{\gamma \in \Gamma} X_\gamma)$.*

Proof. By (16) and Lemma 7.11, for each $\alpha \in (0, 1]$,

$$\overline{[u]_\alpha} = \prod_{\gamma \in \Gamma} \overline{[u_\gamma]_\alpha} = \prod_{\gamma \in \Gamma} [u_\gamma]_\alpha = [u]_\alpha,$$

thus $u \in F_{USC}(\prod_{\gamma \in \Gamma} X_\gamma)$.

$\square$

In the following theorem, we use H to denote the Hausdorff metric on $C(X_\gamma)$ induced by d_γ. We also use H to denote the Hausdorff metric on $C(\prod_{\gamma \in \Gamma} X_\gamma)$ induced by d.

Theorem 7.13. *Let Γ be a set, and for each $\gamma \in \Gamma$, let (X_γ, d_γ) be a metric space. If A_γ and B_γ are elements in $C(X_\gamma)$ for $\gamma \in \Gamma$, then $H(\prod_{\gamma \in \Gamma} A_\gamma, \prod_{\gamma \in \Gamma} B_\gamma) = \sup_{\gamma \in \Gamma} H(A_\gamma, B_\gamma)$.*

Proof. From Lemma 7.11, $\prod_{\gamma \in \Gamma} A_\gamma$ and $\prod_{\gamma \in \Gamma} B_\gamma$ are elements in $C(\prod_{\gamma \in \Gamma} X_\gamma)$.

Note that $d(x, \prod_{\gamma \in \Gamma} B_\gamma) = \sup_{\gamma \in \Gamma} d_\gamma(x_\gamma, B_\gamma)$ for each $x = (x_\gamma)_{\gamma \in \Gamma} \in \prod_{\gamma \in \Gamma} X_\gamma$. Thus

$$\begin{aligned}
H^*(\prod_{\gamma \in \Gamma} A_\gamma, \prod_{\gamma \in \Gamma} B_\gamma) &= \sup_{x \in \prod_{\gamma \in \Gamma} A_\gamma} d(x, \prod_{\gamma \in \Gamma} B_\gamma) \\
&= \sup_{x \in \prod_{\gamma \in \Gamma} A_\gamma} \sup_{\gamma \in \Gamma} d_\gamma(x_\gamma, B_\gamma) \\
&= \sup_{\gamma \in \Gamma} \sup_{x_\gamma \in A_\gamma} d_\gamma(x_\gamma, B_\gamma) \\
&= \sup_{\gamma \in \Gamma} H^*(A_\gamma, B_\gamma).
\end{aligned}$$

35

So
$$H(\prod_{\gamma \in \Gamma} A_\gamma, \prod_{\gamma \in \Gamma} B_\gamma) = \sup_{\gamma \in \Gamma} H(A_\gamma, B_\gamma).$$

$\square$

Now, we give an example to show that there exists a metric space X and $u, v \in F^1_{USC}(X)$ such that $H([u]_\alpha, [v]_\alpha)$ is a non-measurable function of α on $[0,1]$.

Example 7.14. We see $[0, 100] \setminus \{10\}$ as a metric subspace of $\mathbb{R}$. Let $z \in (0,1]$. Define $u^z \in F^1_{USC}([0,100] \setminus \{10\})$ as

$$[u^z]_\alpha = \begin{cases} \{3\}, & \alpha \in [z, 1], \\ \{3\} \cup (10, 10 + \varepsilon], & \alpha = z(1 - \varepsilon), \ 0 < \varepsilon \le 1. \end{cases}$$

Let $z \in (0,1]$. Define $v^z \in F^1_{USC}([0,100] \setminus \{10\})$ as

$$[v^z]_\alpha = \begin{cases} \{73\}, & \alpha \in (z, 1], \\ [71, 81], & \alpha \in [0, z]. \end{cases}$$

Then for $z \in (0,1]$,

$$H([u^z]_\alpha, [v^z]_\alpha) = \begin{cases} 70, & \alpha \in (z, 1], \\ 78, & \alpha = z, \\ 71 - \varepsilon, & \alpha = z(1 - \varepsilon), \ 0 < \varepsilon \le 1, \end{cases} \tag{17}$$

where H is the Hausdorff metric on $C([0, 100] \setminus \{10\})$ induced by the metric on $[0, 100] \setminus \{10\}$.

We see $[0, 9]$ as a metric subspace of $\mathbb{R}$. Define $w \in F([0, 9])$ as $w(t) = 1$ for all $t \in [0, 9]$.

Let A be a non-measurable set in $(0, 1]$.

Let $u := \prod_{z \in [0,1]} u_z$ and let $v := \prod_{z \in [0,1]} v_z$, where

$$u_z = \begin{cases} u^z, & z \in A, \\ w, & z \in [0, 1] \setminus A, \end{cases} \quad v_z = \begin{cases} v^z, & z \in A, \\ w, & z \in [0, 1] \setminus A. \end{cases}$$

Then by Theorem 7.12, u and v are fuzzy sets in $F^1_{USC}(\prod_{z \in [0,1]} X_z)$, where

$$X_z = \begin{cases} [0, 100] \setminus \{10\}, & z \in A, \\ [0, 9], & z \in [0, 1] \setminus A. \end{cases}$$

36

Here we mention that $(\prod_{z\in[0,1]} X_z, d)$ is a metric space with d given by (15).

By Theorem 7.13,

$$
\begin{aligned}
& H([u]_\alpha, [v]_\alpha) \\
&= \sup_{z\in A} H([u^z]_\alpha, [v^z]_\alpha) \vee \sup_{z\in[0,1]\setminus A} H([0,9],[0,9]) \\
&= \sup_{z\in A} H([u^z]_\alpha, [v^z]_\alpha) \\
& \begin{cases} = 78, & \alpha \in A, \\ \leq 71, & \alpha \in [0,1]\setminus A. \end{cases}
\end{aligned}
$$

So $\{\alpha \in [0,1] : H([u]_\alpha, [v]_\alpha) > 73\} = A$, and thus $H([u]_\alpha, [v]_\alpha)$ is a non-measurable function of α on $[0,1]$.

In the sequel, we give some improvements of Propositions 7.1, 7.5 and Appendix A.6, which are the statements on measurability of $H([u]_\alpha, [v]_\alpha)$ presented in [8]. We first prove Theorem 7.15 which is an improvement of Propositions 7.1 and Appendix A.6. Then we show Theorem 7.17 and use it to improve Theorem 7.15 and Proposition 7.5.

Let $v \in F^1_{USC}(X)$ and let $0 \leq \alpha < \beta \leq 1$. The "variation" $w_v(\alpha, \beta)$ is defined as $w_v(\alpha, \beta) := \sup\{H([v]_\xi, [v]_\eta) : \xi, \eta \in (\alpha, \beta]\}$.

Theorem 7.15. *Let $u \in F^1_{USC}(X)$ and let $v \in F^1_{USCG}(X)$. Then $H([u]_\alpha, [v]_\alpha)$ is a measurable function of α on $[0,1]$.*

Proof. The proof is divided into three steps.

Step (I) $H^*([u]_\alpha, [v]_\alpha)$ is a measurable function of α on $[0,1]$.

Let $\xi \in \mathbb{R}$ and let $n \in \mathbb{N}$. Define

$$
S_\xi := \{\alpha \in [0,1] : H^*([u]_\alpha, [v]_\alpha) \geq \xi\},
$$

$$
S_{\xi,n} := S_\xi \cap (\frac{1}{n}, 1].
$$

To show that $H^*([u]_\alpha, [v]_\alpha)$ is a measurable function of α on $[0,1]$, it suffices to show that for each $\xi \in \mathbb{R}$ and $n \in \mathbb{N}$, $S_{\xi,n}$ is a measurable set.

Since $v \in F^1_{USCG}(X)$, from Lemma 6.5 in [8] for each $k = 1, 2, \ldots$, there exist $\frac{1}{n} = \alpha_1^{(k)} < \cdots < \alpha_{l_k}^{(k)} = 1$ such that $w_v(\alpha_i^{(k)}, \alpha_{i+1}^{(k)}) \leq \frac{1}{k}$ for all $i = 1, \ldots, l_k - 1$.

Let $T_{k,i} := \{x : \text{ there exists } s \in S_\xi \text{ such that } \alpha_i^{(k)} < x \leq s \leq \alpha_{i+1}^{(k)}\}$. Put $T_k := \cup_{i=1}^{l_k-1} T_{k,i}$. We affirm that

(i) T_k is a measurable set,

(ii) $T_k \supseteq S_{\xi,n}$, and

(iii) $T_k \subseteq S_{\xi - \frac{1}{k}, n}$.

If $T_{k,i} \neq \emptyset$, then $T_{k,i}$ is an interval. Thus (i) is true. (ii) follows from the definition of T_k.

For each $i = 1, \ldots, l_k - 1$ and each $x \in T_{k,i}$, there exists an $s \in S_\xi$ such that $\alpha_i^{(k)} < x \leq s \leq \alpha_{i+1}^{(k)}$, and thus

$$
\begin{aligned}
&H^*([u]_x, [v]_x) \\
&\geq H^*([u]_s, [v]_x) \\
&\geq H^*([u]_s, [v]_s) - H^*([v]_x, [v]_s) \\
&\geq \xi - 1/k.
\end{aligned}
$$

Hence $T_k \subseteq S_{\xi - \frac{1}{k}}$. Clearly, $T_k \subseteq (\frac{1}{n}, 1]$. So (iii) is proved.

By affirmations (ii) and (iii), we have

$$
S_{\xi,n} \subseteq \bigcap_{k=1}^{+\infty} T_k \subseteq \bigcap_{k=1}^{+\infty} S_{\xi - \frac{1}{k}, n} = S_{\xi,n}. \tag{18}
$$

From affirmation (i), $\bigcap_{k=1}^{+\infty} T_k$ is measurable, and thus by (18), $S_{\xi,n} = \bigcap_{k=1}^{+\infty} T_k$ is measurable.

Step (II) $H^*([v]_\alpha, [u]_\alpha)$ is a measurable function of α on $[0, 1]$.

The proof of Step (II) is similar to that of Step (I).

Let $\xi \in \mathbb{R}$ and let $n \in \mathbb{N}$. Define

$$
S^\xi := \{\alpha \in [0, 1] : H^*([v]_\alpha, [u]_\alpha) \geq \xi\},
$$

$$
S^{\xi,n} := S^\xi \cap (\frac{1}{n}, 1].
$$

To show that $H^*([v]_\alpha, [u]_\alpha)$ is a measurable function of α on $[0, 1]$, it suffices to show that for each $\xi \in \mathbb{R}$ and $n \in \mathbb{N}$, $S^{\xi,n}$ is a measurable set.

Let $T^{k,i} := \{x : \text{ there exists } s \in S^\xi \text{ such that } \alpha_i^{(k)} < s \leq x \leq \alpha_{i+1}^{(k)}\}$. Put $T^k := \cup_{i=1}^{l_k - 1} T^{k,i}$. We affirm that

(i′) T^k is a measurable set,

(ii′) $T^k \supseteq S^{\xi,n}$, and

38

(iii′) $T^k \subseteq S^{\xi-\frac{1}{k},n}$.

(i′) is true because if $T^{k,i} \neq \emptyset$, then $T^{k,i}$ is a point or an interval. (ii′) follows from the definition of T^k.

For each $i = 1, \ldots, l_k - 1$ and each $x \in T^{k,i}$, there exists an $s \in S^\xi$ such that $\alpha_i^{(k)} < s \leq x \leq \alpha_{i+1}^{(k)}$, and thus

$$
\begin{aligned}
& H^*([v]_x, [u]_x) \\
& \geq H^*([v]_x, [u]_s) \\
& \geq H^*([v]_s, [u]_s) - H^*([v]_s, [v]_x) \\
& \geq \xi - 1/k.
\end{aligned}
$$

Hence $T^k \subseteq S_{\xi-\frac{1}{k}}$. Clearly, $T^k \subseteq (\frac{1}{n}, 1]$. So (iii′) is proved.

From affirmations (ii′) and (iii′),

$$
S^{\xi,n} \subseteq \bigcap_{k=1}^{+\infty} T^k \subseteq \bigcap_{k=1}^{+\infty} S^{\xi-\frac{1}{k},n} = S^{\xi,n}. \tag{19}
$$

So by affirmation (i′) and (19), $S^{\xi,n} = \bigcap_{k=1}^{+\infty} T^k$ is measurable.

Step (III) $H([u]_\alpha, [v]_\alpha)$ is a measurable function of α on $[0,1]$.

Since that $H([u]_\alpha, [v]_\alpha) = \max\{H^*([u]_\alpha, [v]_\alpha), H^*([v]_\alpha, [u]_\alpha)\}$, then the desired result follows immediately from the fact that both $H^*([u]_\alpha, [v]_\alpha)$ and $H^*([v]_\alpha, [u]_\alpha)$ are measurable functions of α on $[0,1]$, which is proved in steps (I) and (II).

$\square$

Remark 7.16. Theorem 7.15 is an improvement of Proposition Appendix A.6. Since a singleton set is a compact set, Theorem 7.15 is also an improvement of Proposition 7.1.

Obviously, if $\xi \leq 0$, then $S_\xi = S^\xi = [0,1]$ and $S_{\xi,n} = S^{\xi,n} = [\frac{1}{n}, 1]$.

Let (X, d_X) be a metric subspace of (Y, d_Y). To distinguish from the closure of S in (X, d_X), we use $\overline{S}^Y$ to denote the closure of S in (Y, d_Y).

For $u \in F_{USC}^1(X)$, define $\boldsymbol{u^Y} \in \boldsymbol{F_{USC}^1(Y)}$ as

$$
[u^Y]_\alpha = \cap_{\beta<\alpha} \overline{[u]_\beta}^Y \text{ for } \alpha \in (0,1]. \tag{20}
$$

Note that $[u^Y]_\alpha = \cap_{\beta<\alpha}[u^Y]_\beta$ for all $\alpha \in (0,1]$ (since we can see that for $\alpha \in (0,1]$, $\cap_{\beta<\alpha}[u^Y]_\beta = \cap_{\beta<\alpha} \cap_{\gamma<\beta} \overline{[u]_\gamma}^Y = \cap_{\gamma<\alpha} \overline{[u]_\gamma}^Y = [u^Y]_\alpha$). Then by Theorem 7.10, u^Y is well-defined.

For each $u \in F_{USC}^1(X)$, define

$$\Gamma(u)^Y := \{\alpha \in (0,1] : [u^Y]_\alpha \supsetneqq \overline{[u]_\alpha}^Y\}.$$

If there is no confusion, we will write $\Gamma(u)^Y$ as $\Gamma(u)$ for simplicity.

We use H to denote the Hausdorff metric on $C(X)$ induced by d_X, and we also use H to denote the Hausdorff metric on $C(Y)$ induced by d_Y.

We will use the following Theorem 7.17 to improve Theorem 7.15 and Proposition 7.5.

Theorem 7.17. *Let (X, d_X) be a metric subspace of (Y, d_Y) and let $u, v \in F_{USC}^1(X)$. Then*
(i) $[u^Y]_\alpha \supseteq \overline{[u]_\alpha}^Y$ for all $\alpha \in (0,1]$, and $[u^Y]_0 = \overline{[u]_0}^Y$.
(ii) For each $\alpha \in [0,1] \setminus (\Gamma(u) \cup \Gamma(v))$,

$$H([u^Y]_\alpha, [v^Y]_\alpha) = H([u]_\alpha, [v]_\alpha).$$

(iii) The cardinality of $\Gamma(u)$ is less than the cardinality of $Y \setminus X$.

Proof. (i) follows from the definition of u^Y. The proof is routine. From (20), clearly $[u^Y]_\alpha \supseteq \overline{[u]_\alpha}^Y$. To show $[u^Y]_0 = \overline{[u]_0}^Y$, it suffices to show that $[u^Y]_0 \supseteq \overline{[u]_0}^Y$ and $[u^Y]_0 \subseteq \overline{[u]_0}^Y$. $[u^Y]_0 = \overline{\cup_{\alpha>0}[u^Y]_\alpha}^Y \supseteq \overline{\cup_{\alpha>0}[u]_\alpha}^Y = \overline{[u]_0}^Y$. For each $\alpha \in (0,1]$, $[u^Y]_\alpha \subseteq \overline{[u]_{\alpha/2}}^Y \subseteq \overline{[u]_0}^Y$, and hence $[u^Y]_0 = \overline{\cup_{\alpha>0}[u^Y]_\alpha}^Y \subseteq \overline{[u]_0}^Y$.

From (i) and the definition of $\Gamma(u)$, for each $\alpha \in [0,1] \setminus (\Gamma(u) \cup \Gamma(v))$,

$$H([u^Y]_\alpha, [v^Y]_\alpha) = H(\overline{[u]_\alpha}^Y, \overline{[v]_\alpha}^Y) = H([u]_\alpha, [v]_\alpha),$$

and thus (ii) is proved.

To show that (iii) is true, it suffices to construct an injection $j : \Gamma(u) \to Y \setminus X$.

Let $\gamma \in \Gamma(u)$. Then there is an $x_\gamma \in Y$ such that $x_\gamma \in [u^Y]_\gamma \setminus \overline{[u]_\gamma}^Y$. Define $j(\gamma) = x_\gamma$ for each $\gamma \in \Gamma(u)$. Since $x_\gamma \notin [u]_\gamma = \cap_{\beta<\gamma}[u]_\beta$, there is a $\beta < \gamma$ such that $x_\gamma \notin [u]_\beta$. On the other hand, since $x_\gamma \in [u^Y]_\gamma$, we have $x_\gamma \in \overline{[u]_\beta}^Y$. Thus $x_\gamma \in Y \setminus X$. Hence j is an function from $\Gamma(u)$ to $Y \setminus X$.

Let $\xi, \eta \in \Gamma(u)$ with $\xi < \eta$. Since $x_\xi \notin \overline{[u]_\xi}^Y$, then $x_\xi \notin [u^Y]_\lambda$ when $\lambda > \xi$. Hence $x_\xi \notin [u^Y]_\eta$. Notice that $x_\eta \in [u^Y]_\eta$, and therefore $x_\xi \neq x_\eta$. Thus j is an injection. So (iii) is proved. $\qquad\square$

Corollary 7.18. *Let (X, d_X) be a metric subspace of (Y, d_Y) and $Y \setminus X$ an at most countable set. Then for $u, v \in F^1_{USC}(X)$, $H([u^Y]_\alpha, [v^Y]_\alpha)$ is a measurable function of α on $[0, 1]$ if and only if $H([u]_\alpha, [v]_\alpha)$ is a measurable function of α on $[0, 1]$.*

Proof. By (ii), (iii) of Theorem 7.17, we have that $H([u^Y]_\alpha, [v^Y]_\alpha) = H([u]_\alpha, [v]_\alpha)$ on $[0, 1]$ except at most countable $\alpha \in [0, 1]$. Thus we obtain the desired result.

$\square$

Let $S \subseteq \mathbb{R}^m$. We see $\mathbb{R}^m \setminus S$ as a metric subspace of $\mathbb{R}^m$.

Corollary 7.19. *Let S be an at most countable subset of $\mathbb{R}^m$. Let $u, v \in F^1_{USC}(\mathbb{R}^m \setminus S)$. Then $H([u]_\alpha, [v]_\alpha)$ is a measurable function of α on $[0, 1]$.*

Proof. The desired result follows from Proposition 7.5 and Corollary 7.18. Let $Y = \mathbb{R}^m$. Then from Proposition 7.5, $H([u^Y]_\alpha, [v^Y]_\alpha)$ is a measurable function of α on $[0, 1]$. So by Corollary 7.18, $H([u]_\alpha, [v]_\alpha)$ is a measurable function of α on $[0, 1]$. $\square$

Corollary 7.20. *Let (X, d_X) be a metric subspace of (Y, d_Y) and $Y \setminus X$ an at most countable set. Let $u, v \in F^1_{USC}(X)$. If $u^Y \in F^1_{USCG}(Y)$, then $H([u]_\alpha, [v]_\alpha)$ is a measurable function of α on $[0, 1]$.*

Proof. The desired result follows from Theorem 7.15 and Corollary 7.18. Since $v^Y \in F^1_{USC}(Y)$ and $u^Y \in F^1_{USCG}(Y)$, then by Theorem 7.15, $H([u^Y]_\alpha, [v^Y]_\alpha)$ is a measurable function of α on $[0, 1]$. Thus from Corollary 7.18, $H([u]_\alpha, [v]_\alpha)$ is a measurable function of α on $[0, 1]$.

$\square$

Remark 7.21. Let (X, d_X) be a metric subspace of (Y, d_Y). Clearly, if $u \in F^1_{USCG}(X)$, then $[u]_\alpha = [u^Y]_\alpha$ for $\alpha \in (0, 1]$ and thus $u^Y \in F^1_{USCG}(Y)$. So Corollary 7.20 is an improvement of Theorem 7.15.

Corollary 7.19 is an improvement of Proposition 7.5.

Theorem 7.15 is the special case of Corollary 7.20 when $Y = X$. Proposition 7.5 is the special case of Corollary 7.19 when $S = \emptyset$.

The results in this section was recorded in [12]. In essence, contents including Theorem 7.17, Corollaries 7.18 and 7.19 have already been proved in chinaXiv:202108.00116v1, which is a previous version of [12].

41

8. Conclusions

In this paper, we discuss properties of the d_p metrics and the spaces of fuzzy sets in a general metric space (X, d) with d_p metrics.

We show that for $u, v \in F^1_{USC}(X)$, $d^*_p(u, v) \geq \left(\frac{H_{\mathrm{end}}(u,v)^{p+1}}{p+1} \right)^{1/p}$. If $d_p(u, v)$ is well-defined, then $d_p(u, v) = d^*_p(u, v)$ and of course $d^*_p(u, v)$ can be replaced by $d_p(u, v)$ in the above inequality.

We obtain the characterizations of total boundedness, relative compactness and compactness in $(F^1_{USCG}(X)^p, d_p)$. These conclusions generalize the corresponding conclusions in [10]. Our results indicate that for a subset U in $F^1_{USCG}(X)^p$, U is total bounded (respectively, relatively compact, compact) in $(F^1_{USCG}(X)^p, d_p)$ if and only if U is total bounded (respectively, relatively compact, compact) in $(F^1_{USCG}(X), H_{\mathrm{end}})$ and U is p-mean equi-left-continuous.

We show that $(F^1_{USCG}(\widetilde{X})^p, d_p)$ is a completion of $(F^1_{USCB}(X), d_p)$, and thus a completion of $(F^1_{USCG}(X)^p, d_p)$.

In what cases the d_p metrics are well-defined is a fundamental question. For $u, v \in F^1_{USC}(X)$, $d_p(u, v)$ is well-defined if and only if $H([u]_\alpha, [v]_\alpha)$ is a measurable function of α on $[0, 1]$. In Section 7 of this paper, we obtain the following conclusion.

(i) Let (X, d_X) be a metric subspace of (Y, d_Y) and $Y \setminus X$ an at most countable set. Let $u, v \in F^1_{USC}(X)$. If $u^Y \in F^1_{USCG}(Y)$, then $d_p(u, v)$ is well-defined.

In the special case of $Y = X$, the above conclusion become:

Let $u \in F^1_{USC}(X)$ and let $v \in F^1_{USCG}(X)$. Then $d_p(u, v)$ is well-defined.

(ii) Let S be an at most countable subset of $\mathbb{R}^m$. We see $\mathbb{R}^m \setminus S$ as a metric subspace of $\mathbb{R}^m$. For $u, v \in F^1_{USC}(\mathbb{R}^m \setminus S)$, $d_p(u, v)$ is well-defined.

In the special case of $S = \emptyset$, the above conclusion become:

For $u, v \in F^1_{USC}(\mathbb{R}^m)$, $d_p(u, v)$ is well-defined (we point out this conclusion in [8]).

(iii) There exists a metric space X and $u, v \in F^1_{USC}(X)$ such that $d_p(u, v)$ is not well-defined (we point out this conclusion in [8]). In [8], we introduce the d^*_p metric on $F^1_{USC}(X)$, which is an expansion of the d_p distance on $F^1_{USC}(X)$.

The results of this paper have potential applications in the work relevant to d_p distance on fuzzy sets.

Appendix A.

The contents in this section has been submitted in last version.

For $u, v \in F^1_{USC}(X)$ and $r \in \mathbb{R}$, we use the symbol $\{H(u,v) > r\}$ to denote the set $\{\alpha \in [0,1] : H([u]_\alpha, [v]_\alpha) > r\}$.

Proposition Appendix A.1. *For $u, v \in F^1_{USC}(\mathbb{R}^m)$, $H([u]_\alpha, [v]_\alpha)$ is a measurable function of α on $[0,1]$.*

Proof. We only need to show that for each $r \in \mathbb{R}$, the set $\{H(u,v) > r\}$ is a measurable set.

Step (i) For each $r \in \mathbb{R}$, if $\alpha > 0$ and $\alpha \in \{H(u,v) > r\}$, then there exists $\delta(\alpha) > 0$ such that $[\alpha - \delta(\alpha), \alpha] \subseteq \{H(u,v) > r\}$.

We proceed by contradiction. If for each $\delta > 0$, $[\alpha - \delta, \alpha] \nsubseteq \{H(u,v) > r\}$. Then there exists an increasing sequence $\{\gamma_n\}$ such that $\gamma_n \to \alpha$ and

$$H([u]_{\gamma_n}, [v]_{\gamma_n}) \le r.$$

Given $x \in [u]_\alpha$, then $d(x, [v]_{\gamma_n}) \le H([u]_{\gamma_n}, [v]_{\gamma_n}) \le r$. Therefore there exist $y_n \in [v]_{\gamma_n}$ such that $d(x, y_n) = d(x, [v]_{\gamma_n}) \le r$. Hence there is a subsequence $\{y_{n_i}\}$ of $\{y_n\}$ such that $\{y_{n_i}\}$ converges to $y \in \mathbb{R}^m$. Note that $d(x, y) \le r$ and $y \in \cap [v]_{\gamma_{n_i}} = [v]_\alpha$, so we have $d(x, [v]_\alpha) \le r$.

From the arbitrariness of x, $H^*([u]_\alpha, [v]_\alpha) \le r$. Similarly, we can deduce that $H^*([v]_\alpha, [u]_\alpha) \le r$. Thus $H([u]_\alpha, [v]_\alpha) \le r$, which is a contradiction.

Step (ii) For each $r \in \mathbb{R}$, if $\{H(u,v) > r\} \setminus \{0\} \ne \emptyset$, then $\{H(u,v) > r\} \setminus \{0\}$ is a union of disjoint positive length intervals.

Suppose that $\{H(u,v) > r\} \setminus \{0\} \ne \emptyset$. For $x \in \{H(u,v) > r\} \setminus \{0\}$, let $\overwidehat{x} = \bigcup\{[a,b] : x \in [a,b] \subseteq \{H(u,v) > r\} \setminus \{0\}\}$, i.e. $\overwidehat{x}$ is the largest interval in $\{H(u,v) > r\} \setminus \{0\}$ which contains x. Then by step (i), $\overwidehat{x}$ is a positive length interval. Note that for $x, y \in \{H(u,v) > r\} \setminus \{0\}$, if $\overwidehat{x} \cap \overwidehat{y} \ne \emptyset$, then $\overwidehat{x} = \overwidehat{y}$. Thus $\{H(u,v) > r\} \setminus \{0\}$ is a union of disjoint positive length intervals.

Step (iii) For each $r \in \mathbb{R}$, $\{H(u,v) > r\}$ is a measurable set.

Clearly, if positive length intervals are disjoint, then these positive length intervals are at most countable. Thus, from step (ii), $\{H(u,v) > r\} \setminus \{0\}$ is a measurable set. So $\{H(u,v) > r\}$ is a measurable set. $\square$

Remark Appendix A.2. Let $u, v \in F_{USC}^1(X)$. For each $r \in \mathbb{R}$, if $0 \in \{H(u,v) > r\}$, then there exists $\delta > 0$ such that $[0, \delta] \subseteq \{H(u,v) > r\}$.

The above fact is equivalent to the following fact

Let $u, v \in F_{USC}^1(X)$. Then $H([u]_0, [v]_0) \leq \liminf_{\alpha \to 0+} H([u]_\alpha, [v]_\alpha)$, here $\liminf_{\alpha \to 0+} H([u]_\alpha, [v]_\alpha) = +\infty$ is possible.

Combined this fact with the proof of Proposition Appendix A.1, we have the following conclusion

Let $u, v \in F_{USC}^1(\mathbb{R}^m)$ and let $r \in \mathbb{R}$. If $\{H(u,v) > r\} \neq \emptyset$, then $\{H(u,v) > r\}$ is a union of disjoint positive length intervals (Obviously, $\{H(u,v) > r\}$ could be an interval. It is easy to see that for fixed $r \geq 0$, the possible forms of the maximal intervals in $\{H(u,v) > r\}$ are as follows: $[0, \alpha)$, $[0, \alpha]$, (β, α) and $(\beta, \alpha]$, where $\alpha \in (0, 1]$ and $\beta \in [0, 1)$).

For $f : [0, 1] \to [0, +\infty) \cup \{+\infty\}$ and $\alpha \in (0, 1]$, $\liminf_{\gamma \to \alpha-} f(\gamma)$ is defined by $\liminf_{\gamma \to \alpha-} f(\gamma) := \inf\{x \in [0, +\infty) \cup \{+\infty\} : \text{ there is a sequence } \{\gamma_n\} \text{ such that } \gamma_n \to \alpha - \text{ and } x = \lim_{n \to +\infty} f(\gamma_n)\}$. For $f : [0, 1] \to [0, +\infty) \cup \{+\infty\}$ and $\alpha \in (0, 1]$, $\liminf_{\gamma \to \alpha-} f(\gamma)$ exists and $\liminf_{\gamma \to \alpha-} f(\gamma) = +\infty$ is possible.

We can check that $\liminf_{\gamma \to \alpha-} f(\gamma) = \min\{x \in [0, +\infty) \cup \{+\infty\} : \text{ there is a sequence } \{\gamma_n\} \text{ such that } \gamma_n \to \alpha - \text{ and } x = \lim_{n \to +\infty} f(\gamma_n)\}$. Clearly if $\lim_{\gamma \to \alpha-} f(\gamma)$ exists, then $\lim_{\gamma \to \alpha-} f(\gamma) = \liminf_{\gamma \to \alpha-} f(\gamma)$. In this version, we corrected some misprints in this paragraph and the previous paragraph in the last version.

Remark Appendix A.3. Let $u, v \in F_{USC}^1(X)$ and let $\alpha > 0$. Then the following properties (i) and (ii) are equivalent.
(i) For each $r \in \mathbb{R}$, if $\alpha \in \{H(u,v) > r\}$, then there exists $\delta(\alpha) > 0$ such that $[\alpha - \delta(\alpha), \alpha] \subseteq \{H(u,v) > r\}$.
(ii) $H([u]_\alpha, [v]_\alpha) \leq \liminf_{\gamma \to \alpha-} H([u]_\gamma, [v]_\gamma)$ ($\liminf_{\gamma \to \alpha-} H([u]_\gamma, [v]_\gamma) = +\infty$ is possible).

So for $u, v \in F_{USC}^1(X)$, the property "$H([u]_\alpha, [v]_\alpha) \leq \liminf_{\gamma \to \alpha-} H([u]_\gamma, [v]_\gamma)$ for all $\alpha \in (0, 1]$" is equivalent to the property listed below, which is given as the conclusion of step (i) of the proof of Proposition Appendix A.1:

- For each $r \in \mathbb{R}$, if $\alpha > 0$ and $\alpha \in \{H(u,v) > r\}$, then there exists $\delta(\alpha) > 0$ such that $[\alpha - \delta(\alpha), \alpha] \subseteq \{H(u,v) > r\}$.

The equivalence of properties (i) and (ii) given at the beginning of this remark follows from basic analysis. Assume that (i) is true. Given $r \in \mathbb{R}$ with $H([u]_\alpha, [v]_\alpha) > r$. Then there exists $\delta(\alpha) > 0$ such that $[\alpha - \delta(\alpha), \alpha] \subseteq$

44

$\{H(u,v) > r\}$, thus $\liminf_{\gamma \to \alpha-} H([u]_\gamma, [v]_\gamma) \geq r$. From the arbitrariness of $r \in (-\infty, H([u]_\alpha, [v]_\alpha))$, we have $H([u]_\alpha, [v]_\alpha) \leq \liminf_{\gamma \to \alpha-} H([u]_\gamma, [v]_\gamma)$. So (ii) is true.

Assume that (ii) is true. Let $\alpha > 0$ and $r \in \mathbb{R}$. If $\alpha \in \{H(u,v) > r\}$, then $\liminf_{\gamma \to \alpha-} H([u]_\gamma, [v]_\gamma) > r$. We claim that there exists $\delta(\alpha) > 0$ such that $[\alpha - \delta(\alpha), \alpha] \subseteq \{H(u,v) > r\}$. Otherwise there exists $\{\alpha_n\}$ in $[0,1]$ such that $\alpha_n \to \alpha-$ and $H([u]_{\alpha_n}, [v]_{\alpha_n}) \leq r$, which contradicts $\liminf_{\gamma \to \alpha-} H([u]_\gamma, [v]_\gamma) > r$.

Remark Appendix A.4. We can see that step (i) in the proof of Proposition Appendix A.1 shows the following statement

(a) For $u,v \in F_{USC}^1(\mathbb{R}^m)$, u,v satisfy the property which is given as the conclusion of step (i) in the proof of Proposition Appendix A.1.

We can see that steps (ii) and (iii) in the proof of Proposition Appendix A.1 show the following statement

(b) For $u,v \in F_{USC}^1(X)$, if u,v satisfy the property which is given as the conclusion of step (i) in the proof of Proposition Appendix A.1, then $H([u]_\alpha, [v]_\alpha)$ is a measurable function of α on $[0,1]$.

So from the proof of Proposition Appendix A.1 and Remark Appendix A.3, we know that

(c) For $u,v \in F_{USC}^1(X)$, if $H([u]_\alpha, [v]_\alpha) \leq \liminf_{\gamma \to \alpha-} H([u]_\gamma, [v]_\gamma)$ for all $\alpha \in (0,1]$, then $H([u]_\alpha, [v]_\alpha)$ is a measurable function of α on $[0,1]$.

(a′) Let $u,v \in F_{USC}^1(\mathbb{R}^m)$. Then $H([u]_\alpha, [v]_\alpha) \leq \liminf_{\gamma \to \alpha-} H([u]_\gamma, [v]_\gamma)$ for all $\alpha \in (0,1]$.

The following Proposition Appendix A.5 is Lemma 4.4 in [8].

Proposition Appendix A.5. *[8] Let $U_n \in K(X)$ for $n = 1,2,\ldots$.*
(i) If $U_1 \supseteq U_2 \supseteq \ldots \supseteq U_n \supseteq \ldots$, then $U = \bigcap_{n=1}^{+\infty} U_n \in K(X)$ and $H(U_n, U) \to 0$ as $n \to +\infty$.
(ii) If $V_1 \subseteq V_2 \subseteq \ldots \subseteq V_n \subseteq \ldots$ and $V = \overline{\bigcup_{n=1}^{+\infty} V_n} \in K(X)$, then $H(V_n, V) \to 0$ as $n \to +\infty$.

Proof. This Proposition is Proposition 7.6. □

From Proposition Appendix A.5, we know that

45

Proposition Appendix A.6. *For $u, v \in F^1_{USCG}(X)$, $H([u]_\alpha, [v]_\alpha)$ is a measurable function of α on $[0, 1]$.*

Proof. Note that $H([u]_\alpha, [v]_\alpha)$ is finite at $\alpha \in (0, 1]$ and for $\alpha, \beta \in (0, 1]$,

$$|H([u]_\alpha, [v]_\alpha) - H([u]_\beta, [v]_\beta)| \le H([u]_\alpha, [u]_\beta) + H([v]_\alpha, [v]_\beta).$$

Then by Proposition Appendix A.5 (i), for each $\alpha \in (0, 1]$,

$$\lim_{\beta \to \alpha-} H([u]_\beta, [v]_\beta) = H([u]_\alpha, [v]_\alpha),$$

i.e. $H([u]_\alpha, [v]_\alpha)$ is left-continuous at $\alpha \in (0, 1]$.

Thus from clause (c) in Remark Appendix A.4, we have $H([u]_\alpha, [v]_\alpha)$ is a measurable function of α on $[0, 1]$.

The desired result can also be shown in the following way. Since $H([u]_\alpha, [v]_\alpha)$ is left-continuous at $\alpha \in (0, 1]$, then obviously u, v satisfy the property which is given as the conclusion of step (i), and thus by clause (b) in Remark Appendix A.4, $H([u]_\alpha, [v]_\alpha)$ is a measurable function of α on $[0, 1]$.

$\square$

Remark Appendix A.7. From Proposition Appendix A.5, we know that for $u, v \in F^1_{USCG}(X)$, $H([u]_\alpha, [v]_\alpha)$ is left-continuous at $\alpha \in (0, 1]$, and that for $u, v \in F^1_{USCB}(X)$, $H([u]_\alpha, [v]_\alpha)$ is right-continuous at $\alpha = 0$.

The first conclusion mentioned above has been shown in the proof of Proposition Appendix A.6. The following proof of the last conclusion is similar as that of the first conclusion. Since for $u, v \in F^1_{USCB}(X)$ and $\alpha \in (0, 1]$,

$$|H([u]_\alpha, [v]_\alpha) - H([u]_0, [v]_0)| \le H([u]_\alpha, [u]_0) + H([v]_\alpha, [v]_0).$$

Thus by Proposition Appendix A.5 (ii), $\lim_{\alpha \to 0+} H([u]_\alpha, [v]_\alpha) = H([u]_0, [v]_0)$, i.e. $H([u]_\alpha, [v]_\alpha)$ is right-continuous at $\alpha = 0$.

Remark Appendix A.8. In [10] (Lemma 6.3) and [8] (Lemma 6.5), we pointed out that for $u \in F^1_{USCG}(X)$, the cut-function $[u](\alpha) = [u]_\alpha$ from $[0, 1]$ to $(C(X), H)$ is left-continuous on $(0, 1]$. Then it follows immediately that for $u, v \in F^1_{USCG}(X)$, $H([u]_\alpha, [v]_\alpha)$ is left-continuous at $\alpha \in (0, 1]$ (see Proposition Appendix A.6). From this fact, it's natural to realize that for $u, v \in F^1_{USCG}(X)$, $H([u]_\alpha, [v]_\alpha)$ is a measurable function of α on $[0, 1]$.

We suspect the conclusion that a left-continuous function $f : (0,1] \to \mathbb{R}$ is measurable is an already known conclusion, although we can't find this conclusion in the references that we can obtain. So in some sense, Proposition Appendix A.6 can be seen as a corollary of Proposition Appendix A.5.

Let (X, d_X) be a metric space. We say that $S \subseteq F_{USC}^1(X)$ satisfies condition (X, d_X)-I if $[u]_\alpha \cap B(x, r)$ is compact in (X, d_X) for all $u \in S$, $\alpha \in (0, 1]$, $x \in X$ and $r \in \mathbb{R}^+$, where $B(x, r) := \{y \in X : d_X(x, y) \le r\}$.

Clearly, $S = F_{USC}^1(\mathbb{R}^m)$ satisfies condition $\mathbb{R}^m$-I and $S = F_{USCG}^1(X)$ satisfies condition (X, d_X)-I.

If $S \subseteq F_{USC}^1(X)$ satisfies condition (X, d_X)-I, then proceed similarly as the step (i) of the proof of Proposition Appendix A.1, we have that $H([u]_\alpha, [v]_\alpha) \le \liminf_{\gamma \to \alpha-} H([u]_\gamma, [v]_\gamma)$ for all $u, v \in S$ and $\alpha \in (0, 1]$. Thus as mentioned in Remark Appendix A.4, for all $u, v \in S$, $H([u]_\alpha, [v]_\alpha)$ is a measurable function of α on $[0, 1]$.

There exists metric space (X, d_X) and $S \subseteq F_{USC}^1(X)$ which satisfies a condition weaker than condition (X, d_X)-I. By using this weaker condition, we can proceed similarly as the step (i) of the proof of Proposition Appendix A.1 to show that $H([u]_\alpha, [v]_\alpha) \le \liminf_{\gamma \to \alpha-} H([u]_\gamma, [v]_\gamma)$ for all $u, v \in S$ and $\alpha \in (0, 1]$.

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